

# Non archimedean gauge seminorms

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## Abstract

We fix a base commutative topological ring  $k$ , separated and complete in a linear topology. Within the category  $\mathcal{LM}_k$  of  $k$ -linearly topologized  $k$ -modules, we single-out the full subcategory  $\mathcal{LM}_k^u$  of  $k$ -modules whose scalar product is uniformly continuous. We describe limits and colimits, and introduce a tensor product  $\widehat{\otimes}_k^c$  (resp.  $\widehat{\otimes}_k^u$ ) in  $\mathcal{LM}_k$  (resp. in  $\mathcal{LM}_k^u$ ). When  $k = K^\circ$ , for a non trivially valued non archimedean field  $K$ ,  $K$ -Banach spaces [12] are objects of  $\mathcal{LM}_k$  but not of  $\mathcal{LM}_k^u$ . We propose a definition of a *pseudobanach*  $k$ -module which coincides with the one of a  $K$ -Banach space if  $k = K^\circ$ , but covers in general the notion of a family of Banach spaces over variable fields. We describe the category  $\mathcal{LR}_k$  (resp.  $\mathcal{RR}_k$ ) of complete  $k$ -linearly (resp. linearly) topologized  $k$ -rings and the full subcategory  $\mathcal{LR}_k^u$  of  $\mathcal{LR}_k$  of the  $k$ -rings for which the scalar product is uniformly continuous. We discuss limits and colimits in  $\mathcal{LR}_k$  (resp.  $\mathcal{LR}_k^u$ , resp.  $\mathcal{RR}_k$ ) and examine their commutation with the monoidal structures  $\widehat{\otimes}_k^c$  (resp.  $\widehat{\otimes}_k^u$ ). The former monoidal structure is analog to both Schneider's  $-\widehat{\otimes}_{K,\iota}-$  and  $-\widehat{\otimes}_{K,\pi}-$  [12], while the latter is the one used in the theory of formal schemes.

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## 0 Introduction

The main motivation of this paper is that of providing solid foundations to the theory of commutative group and ring functors on categories of commutative topological  $k$ -rings, over some fixed base ring  $k$  which is complete in a linear topology. In particular, the present paper is preliminary to [1].

Recent developments of arithmetic geometry, in particular, use variants W, CW, BW of the functors of Witt vectors, covectors and bivectors defined for topological rings of characteristic  $p > 0$  [2], [7], [9], to establish the remarkable *tilting equivalence* of Scholze [13]. Such Witt-type functors can also be globalized so that to apply to (non-archimedean) analytic spaces and to extend geometrically the previous equivalence to relative situations [10], [11]. The discussion of the most general type of topological rings to which these functors apply is however usually avoided. In general, one considers Banach rings rather than rings complete in a linear topology. On the other hand, a glance at the literature indicates that little more than definitions are to be found about the category  $\mathcal{LR}_k$  (resp.  $\mathcal{RR}_k$ ) of  $k$ -rings which are complete in a  $k$ -linear (resp. linear) topology, as soon as one leaves the safe continent of Noetherian adic rings or of mild variations of such.

From another viewpoint, a rich theory of locally convex topological vector spaces over a non-archimedean field  $K$  exists [12], and the correspondence

$$\text{open lattice} \longleftrightarrow \text{gauge seminorm}$$

establishes a link between the additional information provided by a seminorm and the one obtained by regarding a topological  $K$ -vector space as linearly topologized complete  $k$ -module, where  $k = K^\circ$ , the ring of integers of  $K$ . We point out however that such rings  $k$  are quite special. In particular they are essentially one-dimensional, while we are interested in higher-dimensional base-rings, as well.

We are lead to single-out within the category  $\mathcal{LM}_k$  of  $k$ -linearly topologized separated and complete topological  $k$ -modules  $M$ , for which the scalar product  $k \times M \rightarrow M$  is continuous for the product topology, the full subcategory  $\mathcal{LM}_k^u$  consisting of *uniform* objects, namely those  $M$  for which the scalar product is *uniformly* continuous for the product uniformity of  $k \times M$ . Notice that, unless  $K$  is trivially valued, a non-zero  $K$ -Banach space is never a uniform object of  $\mathcal{LM}_k$ , for  $k = K^\circ$ . On the other hand, a standard assumption in the theory of, say,  $k$ -formal schemes topologically locally of finite type over a Noetherian  $k$ , is that the  $k$ -linear topologies on any  $k$ -module  $M$  considered should be weaker than the topology induced by  $k$  (meaning the *naive canonical topology* of  $M$ , Definition 2.4). This is precisely the meaning of  $M$  being uniform, see Lemma 2.5. Such an assumption cannot be made in our context since it does not generally hold for a non-archimedean field  $K$  itself, on which  $k = K^\circ$  induces the trivial topology, namely  $\{\emptyset, K\}$ . Moreover we cannot make any finiteness assumption, and  $k$  in particular may not be Noetherian.

The existence of the full subcategory  $\mathcal{LM}_k^u$  of  $\mathcal{LM}_k$  generates two distinct notions  $\widehat{\otimes}_k^u$  and  $\widehat{\otimes}_k^c$  of topological tensor product, where the apex “ $u$ ” refers to “uniform” while the apex “ $c$ ” refers to “continuous”. The monoidal structure  $\widehat{\otimes}_k^c$  is the completion of the one used for Fréchet spaces in [12], and there denoted  $\otimes_{K,\pi} = \otimes_{K,\iota}$ , while  $\widehat{\otimes}_k^u$  is the monoidal structure used in the theory of  $k$ -formal groups topologically of finite type.

The purpose of this paper is twofold. On the one hand, we want to encompass the theory of Banach vector spaces over a non archimedean field  $K$ , and of continuous  $K$ -linear maps, within the theory of  $k$ -linearly topologized separated and complete topological  $k$ -modules,

where  $k = K^\circ$ . In particular, we want to characterize  $K$ -Banach spaces within such topological  $K^\circ$ -modules, with no reference to a norm. Notice that  $k$  is a linearly topologized ring in the sense that a fundamental system of neighborhoods of 0 in  $k$  consists of open ideals. We obtain the notion of a *pseudobanach  $k$ -module* or  *$k$ -algebra* (see Definition 4.1 and Definition 11.1 below). When  $k = K^\circ$  for a non trivially valued complete non-archimedean field, the full subcategory  $\mathcal{PB}_k$  (resp.  $\mathcal{PBA}_k$  (resp.  $\mathcal{UPBA}_k$ )) of  $\mathcal{LM}_k$  (resp.  $\mathcal{LR}_k$ ) of pseudobanach  $k$ -modules (resp. of commutative pseudobanach  $k$ -algebras (resp. of pm-type, see Definition 11.1 below)) is equivalent to the category of  $K$ -Banach spaces (resp. of commutative  $K$ -Banach algebras (resp. of pm-type [7])) and continuous  $K$ -linear homomorphisms. This part of our discussion may be seen as a non-archimedean analog of the theory of gauge seminorms, as Schneider's [12], with the difference that it is developed over  $k$ , with no reference to  $K$ . This, by the way, accounts for the title we chose for this paper. For general  $k$ , the categories  $\mathcal{PB}_k$ ,  $\mathcal{PBA}_k$  and  $\mathcal{UPBA}_k$  are new.

The monoidal structure  $-\hat{\otimes}_k^c C$  is used to define base change for  $K$ -Banach algebras (viewed as  $k = K^\circ$ -modules) via a morphism  $k \rightarrow C$  of  $\mathcal{RR}_k$ . The first main result of this paper is Proposition 7.6 which shows that, under the mild condition **OPW** on  $k$  and  $C$ , if  $C/k$  is pro-flat (see Definition 7.2), then for any  $k$ -pseudobanach space  $M$ ,  $(M)_C^c = M \hat{\otimes}_k^c C$  is a  $C$ -pseudobanach space.

On the other hand, for application to the representability of our group and ring functors we need to have at our disposal a full subcategory  $\mathcal{IRR}_k$  of  $\mathcal{LR}_k$  containing the algebras representing all functors of interest to us, together with a base-change functor  $(-)_C^{\text{ind}} : \mathcal{IRR}_k \rightarrow \mathcal{IRR}_C$  for  $k \rightarrow C$  as before. To give an idea of the difficulties we faced, we point out that our group or ring functors are defined on the entire category  $\mathcal{LR}_k$ . They are represented by Hopf or bi-Hopf algebra objects of  $\mathcal{LR}_k^u := \mathcal{LR}_k \cap \mathcal{LM}_k^u$  which however are *inductive limits in  $\mathcal{LR}_k$*  (hence also in  $\mathcal{LR}_k^u$ ), denoted  $\varinjlim_\alpha {}^\mathcal{L}R_\alpha$  (resp.  $\varinjlim_\alpha {}^uR_\alpha$ ), of *inductive systems  $\{R_\alpha\}_\alpha$  in  $\mathcal{RR}_k$* . This property defines the full subcategory  $\mathcal{IRR}_k$  of  $\mathcal{LR}_k$ . The basic linear algebra constructions involved require special attention and raise the general problem of existence and description of limits and colimits, and of suitable monoidal structures. In particular, assume one of our functors  $F$  is represented by an object  $\mathcal{L}_k = \varinjlim_\alpha {}^\mathcal{L}R_\alpha$  in  $\mathcal{LR}_k^u$ , as before, in the sense that

$$F(X) = \text{Hom}_{\mathcal{LR}_k}(\mathcal{L}_k, X),$$

for any  $X$  in  $\mathcal{LR}_k$ . Then the second main result of this paper, Proposition 8.4, shows that the restriction of  $F$  to  $\mathcal{LR}_C^u$ , where  $C$  is an object of  $\mathcal{RR}_k$ , is represented by an object  $\mathcal{L}_C := (\mathcal{L}_k)_C^{\text{ind}}$  of  $\mathcal{LR}_C^u$  which is also an inductive limit, namely  $\varinjlim_\alpha {}^\mathcal{L}(R_\alpha)_C^u$ , in  $\mathcal{LR}_C$  of an inductive system in  $\mathcal{RR}_C$ . Here, for  $X$  in  $\mathcal{RR}_k$ ,  $(X)_C^u = X \hat{\otimes}_k^u C$  is an object of  $\mathcal{RR}_C$ , see (6.0.1) and (3.3.2). This at least shows that the functor

$$\begin{aligned} (-)_C^{\text{ind}} : \mathcal{IRR}_k &\longrightarrow \mathcal{IRR}_C \\ \mathcal{L}_k &\longmapsto \mathcal{L}_C = (\mathcal{L}_k)_C^{\text{ind}} \end{aligned}$$

is well-defined (see Remark 8.5 below).

It will be shown in [1] that the colimit  $\mathcal{T}(\mathcal{L}_k) = \varinjlim_\alpha {}^\mathcal{R}R_\alpha$  (see Lemma 5.7), *taken this time in  $\mathcal{RR}_k$* , of the same inductive system  $\{R_\alpha\}_\alpha$  in  $\mathcal{RR}_k$  which defines  $\mathcal{L}_k$ , represents, under suitable conditions, the subfunctor  $X \mapsto F^{\text{bd}}(X)$  of bounded elements of  $F(X)$ . The importance of finding a topological algebra representing the subfunctors  $W^{\text{bd}}$ ,  $CW^{\text{bd}}$  and  $BW^{\text{bd}}$  of [7], [9] can hardly be overestimated.

For applications it is important to consider the restriction to  $\mathcal{UPBA}_k$  of functors  $F$  of the type described above. We recall that, when  $k = K^\circ$ , as above,  $\mathcal{UPBA}_k$  identifies with

the category of  $K$ -Banach algebras of pm-type of [7]. In [1] we will define the functors  $CW$ ,  $BW$ ,  $W^{\text{bd}}$ ,  $CW^{\text{bd}}$  and  $BW^{\text{bd}}$  on  $\mathcal{LR}_k$  when  $k$  is of characteristic  $p$  and is *topologically perfect*, in the sense that its Frobenius is an automorphism. We will prove that  $CW^{\text{bd}}$  (resp.  $BW^{\text{bd}}$ ) induces a functor  $\mathcal{UPBA}_k \rightarrow \mathcal{PBA}_{W(k)}$  (resp.  $\mathcal{UPBA}_k \rightarrow \mathcal{UPBA}_{W(k)}$ ). Of special interest is the case of  $k = \kappa = \mathbb{F}_p[[t^{p^{-\infty}}]]$ , completed in the  $t$ -adic topology. In that case, for  $R$  an object of  $\mathcal{UPBA}_\kappa$ ,  $CW_\kappa^{\text{bd}}(R)$  (resp.  $BW_\kappa^{\text{bd}}(R)$ , for  $R$  topologically perfect) is an object of  $\mathcal{PBA}_{W(\kappa)}$  (resp.  $\mathcal{UPBA}_{W(\kappa)}$ ), where  $W(\kappa)$  is of dimension 2. Then, for any quotient map  $W(\kappa) \rightarrow L^\circ$  onto the ring of integers of a perfectoid  $p$ -adic field  $L$ ,  $(CW_\kappa^{\text{bd}}(R))_{L^\circ}^c$  (resp.  $(BW_\kappa^{\text{bd}}(R))_{L^\circ}^c$ ) (see Proposition 7.6 (resp. Corollary 11.10) below) becomes an  $L$ -Banach space (resp. a perfectoid algebra over  $L$  [13]) in the usual sense.

The generalities we develop in this paper suffice for the purpose we have in mind, and in particular provide a natural framework to [1]. It appears however that the discussion of linear topologies, and of “linear uniformities”, should be developed more systematically, especially as far as limits and colimits are concerned. It seems that such a systematic discussion does not exist in the literature yet.

*Acknowledgments.* I am very grateful to Maurizio Cailotto for generously dedicating a lot of his time to discuss with me a variety of mathematical questions which arose during the preparation of this article. I had the privilege of discussing the content of this paper with Peter Schneider; his criticism was very useful.

## 1 Notation

A prime number  $p$  is fixed throughout this paper even though it only appears in examples; then  $\mathbb{Z}_p$  and  $\mathbb{Q}_p$  have the usual meaning. Unless otherwise specified, a *ring* is meant to be commutative with 1. We denote by  $\mathcal{Ab}$  (resp.  $\mathcal{Rings}$ , resp.  $\mathcal{Mod}_k$ ) the category of abelian groups (resp. of commutative rings with 1, resp. of unitary  $k$ -modules for  $k$  a ring). Generally speaking, for any ring  $R$  in  $\mathcal{Rings}$ , the  $R$ -algebras appearing in this paper will be understood to be commutative with 1. All the non-archimedean fields  $K = (K, v_K)$  we consider will be complete but not necessarily non trivially valued. We set  $K^\circ := \{x \in K \mid v_K(x) \geq 0\}$  and  $K^{\circ\circ} := \{x \in K \mid v_K(x) > 0\}$ .

## 2 Linearly topologized modules

A topological ring  $k$  is *linearly topologized* if it has a basis of open neighborhoods of 0 consisting of ideals. This implies that the product map

$$\mu_k : k \times k \longrightarrow k, \quad (x, y) \longmapsto xy$$

is uniformly continuous. All over this paper,  $k$  will be a complete separated linearly topologized topological ring. We often call  $\mathcal{P}(k)$  a fundamental system of open ideals of  $k$ . For certain constructions, we will need the further assumption of *openness* on a linearly topologized ring  $R$ , namely

**(OP)** For any regular element  $a$  of  $R$ , the map

$$R \longrightarrow R, \quad x \longmapsto ax$$

is open.

**Remark 2.1.** A typical example of a linearly topologized ring  $R$  for which property **OP** fails is  $\mathbb{Z}[T]$  equipped with the  $(p, T)$ -adic topology. It is clear in fact that the ideals  $(T^N)$  and  $(p^N)$  are not open in  $\mathbb{Z}[T]$ , for any  $N \in \mathbb{Z}_{\geq 1}$ . The condition is instead verified by any complete rank one valuation ring.

**Remark 2.2.** It follows from assumption **OP** that, if  $R$  is a domain, the product map

$$\mu_R : R \times R \longrightarrow R, \quad (x, y) \longmapsto xy$$

is open. In fact, for any open ideals  $I, J$  of  $R$ ,  $IJ = \bigcup_{a \in I - \{0\}} aJ$  is a union of open subsets of  $R$ , hence is open. Now, for any open subsets  $U, V \subset R$ ,  $U = \bigcup_{a \in U} a + I_a$ ,  $V = \bigcup_{a \in V} a + J_a$  where  $I_a, J_a$  are open ideals of  $R$ . Then

$$UV = \bigcup_{a \in U, b \in V} (a + I_a)(b + J_b) = \bigcup_{a \in U, b \in V} ab + bI_a + aJ_b + I_aJ_b$$

is a union of open subsets of  $R$ .

A weaker assumption of openness on a linearly topologized ring  $R$  is

**(OPW)** *For any open ideals  $I$  and  $J$  of  $R$ , the ideal  $IJ$  is open.*

Assumption **OPW** holds for any adic ring [6, Chap. 0, Def. 7.1.9]. Whenever assumption **OP** or **OPW** will be needed, the reader will be explicitly warned. Assumptions **OP** and **OPW** certainly hold for  $R$  if  $R$  is endowed with the discrete topology. Let  $R$  be a linearly topologized ring. A *topological  $R$ -module* is a topological abelian group which is also an  $R$ -module such that the scalar product map

$$R \times M \longrightarrow M, \quad (a, x) \longmapsto ax$$

is continuous *for the product topology of  $R \times M$* . An  *$R$ -linear topology* on an  $R$ -module  $M$  is an  $R$ -module topology which has a basis of open neighborhoods of 0 consisting of  $R$ -submodules of  $M$ . We often call  $\mathcal{P}(M)$  a fundamental system of open  $R$ -modules in  $M$ . Then  $M$  is equipped with a canonical uniform structure and we say that it is *uniform* if the scalar product map is uniformly continuous for the product uniformity of  $R \times M$ . Notice that if the topology of  $k$  is the discrete one, then any topological  $R$ -module is uniform. A complete  $k$ -linearly topologized  $k$ -module  $M$  is meant to be separated. Similarly, when we refer to “completion” we always mean “separated completion”. A relevant condition for an  $R$ -linearly topologized  $R$ -module  $M$  is

**(OPM)** *For any regular element  $a$  of  $R$ , the map*

$$M \longrightarrow M, \quad x \longmapsto ax$$

*is open.*

When discussing topological vector spaces  $V$  over a non archimedean field  $K$ , we will generally choose  $k = K^\circ$ , equipped with the subspace topology and will assume that  $V$  is equipped with a  $k$ -linear topology. In this case  $k$  satisfies **OP** and  $V$  satisfies **OPM** for  $R = k$ . Let  $K$  be non-trivially valued let  $k = K^\circ$ . Let  $V$  be a locally convex  $K$ -vector space in the sense of [12], then the  $k$ -linearly topologized  $k$ -module  $V$  is a topological  $k$ -module as well [12, Lemma 4.1], but it is not uniform, in general. Mostly we think of  $k = \mathbb{F}_p$ , the field with  $p$  elements, or  $= \mathbb{Z}_{(p)}$ , either one equipped with the discrete topology, or of  $k = \mathbb{Z}_p$  equipped with the  $p$ -adic topology.

**Definition 2.3.** *We let  $\mathcal{LM}_k$  (resp.  $\mathcal{LM}_k^u$ ) be the category of complete  $k$ -linearly topologized (resp. uniform) topological  $k$ -modules and continuous  $k$ -linear maps.*

We denote by  $k^{\text{for}}$  the ring underlying the topological ring  $k$ . In general  $M \mapsto M^{\text{for}}$  will be the natural forgetful functor  $\mathcal{LM}_k \longrightarrow \mathcal{Mod}_{k^{\text{for}}}$ . To avoid excessively burdening the notation however, the category  $\mathcal{Mod}_{k^{\text{for}}}$  will be simply denoted by  $\mathcal{Mod}_k$ . Similarly, we generally write  $\text{Hom}_k$  for  $\text{Hom}_{k^{\text{for}}}$ ,  $\text{Bil}_k$  (standing for “ $k$ -bilinear”) for  $\text{Bil}_{k^{\text{for}}}$ , and shorten  $M^{\text{for}} \otimes_{k^{\text{for}}} N^{\text{for}}$  into  $M \otimes_k N$ . (Complete tensor product will have a distinguished notation, anyhow.)

**Definition 2.4.** The naive canonical topology on a  $k$ -module  $M$  is the  $k$ -linear topology with a basis of open  $k$ -submodules consisting of  $\{IM\}_I$ , for  $I$  running over the set of open ideals of  $k$ . For any  $M$  in  $\mathcal{M}od_k$ , we define the object  $M^{\text{can}}$  of  $\mathcal{LM}_k$  to be the completion of  $M$  in its naive canonical topology, i.e. the  $k$ -module

$$(2.4.1) \quad \widehat{M} = \varprojlim_{I \in \mathcal{P}(k)} M/IM,$$

equipped with the weak topology of the projections to the discrete  $k/I$ -modules  $M/IM$ .

**Lemma 2.5.** Let  $M$  be an object of  $\mathcal{LM}_k$ . Then  $M$  is uniform if and only if its topology is weaker than the naive canonical topology.

*Proof.* Assume  $M$  is uniform. Then, for any  $U \in \mathcal{P}(M)$  there is a  $V \in \mathcal{P}(M)$  and an  $I \in \mathcal{P}(k)$  such that for any  $a \in k$  and  $m \in M$ ,

$$(a + I)(m + V) \subset am + U.$$

But this implies that  $IM \subset U$ , so that the topology of  $M$  is weaker than the canonical one. The converse is clear.  $\square$

**Corollary 2.6.** For any  $M$  in  $\mathcal{M}od_k$ ,  $M^{\text{can}}$  is an object of  $\mathcal{LM}_k^u$ .

**Definition 2.7.** The objects of  $\mathcal{LM}_k^u$  of the form  $M^{\text{can}}$ , for an  $M$  in  $\mathcal{M}od_k$ , are said to be canonical (resp.  $k$ -canonical, for more precision) or to have the (resp.  $k$ -)canonical topology.

**Remark 2.8.** Notice that the naive canonical topology on a  $k$ -module  $M$  runs, in general, into a serious difficulty. Namely, it is not true in general that the completion of  $(M, \{IM\}_I)$ , that is the object  $\widehat{M}$  defined above, would still carry the naive canonical topology. We are indebted to Peter Schneider for pointing out this problem.

**Proposition 2.9.** The functors

$$(2.9.1) \quad \begin{array}{ccc} & \xrightarrow{\text{can}} & \\ \mathcal{M}od_k & \xleftarrow{\text{for}} & \mathcal{LM}_k^u \end{array}$$

are adjoint : for any  $M$  in  $\mathcal{M}od_k$  and  $N$  in  $\mathcal{LM}_k^u$ , there are canonical identifications

$$(2.9.2) \quad \text{Hom}_{\mathcal{LM}_k}(M^{\text{can}}, N) = \text{Hom}_{\mathcal{M}od_k}(M, N^{\text{for}}).$$

*Proof.* For any  $N$  in  $\mathcal{LM}_k^u$  we have a canonical continuous  $k$ -linear map

$$(2.9.3) \quad (N^{\text{for}})^{\text{can}} \longrightarrow N.$$

So, for any  $M$  in  $\mathcal{M}od_k$  and  $k$ -module morphism  $f : M \rightarrow N^{\text{for}}$ , we get  $\widehat{f} : \widehat{M} \rightarrow \widehat{N^{\text{for}}}$ , hence a morphism  $f^{\text{can}} : M^{\text{can}} \rightarrow (N^{\text{for}})^{\text{can}} \rightarrow N$ . Conversely, from  $g : M^{\text{can}} \rightarrow N$  we obtain  $M \rightarrow (M^{\text{can}})^{\text{for}} \xrightarrow{g^{\text{for}}} N^{\text{for}}$ .  $\square$

**Lemma 2.10.**

1. The category  $\mathcal{LM}_k$  admits limits. Its full subcategory  $\mathcal{LM}_k^u$  is closed by limits.
2. The category  $\mathcal{LM}_k$  admits colimits.
3. The category  $\mathcal{LM}_k^u$  admits colimits.

4. For a finite inductive system in  $\mathcal{LM}_k^u$ , the two colimits in  $\mathcal{LM}_k^u$  and in  $\mathcal{LM}_k$  coincide.

*Proof.* 1. Let  $(M_\alpha)_{\alpha \in A}$  be a projective system in  $\mathcal{LM}_k$  indexed by the preordered set  $A$ . Its projective limit in  $\mathcal{LM}_k$  is simply the projective limit  $M' = \varprojlim_{\alpha \in A} M_\alpha^{\text{for}}$  equipped with

the weak topology  $\tau$  of the canonical projections  $\pi_\alpha : M' \rightarrow M_\alpha$ . In fact, let us show first that  $(M', \tau)$  is an object of  $\mathcal{LM}_k$ . We pick  $a \in k$ , an element  $m = (m_\alpha)_\alpha \in M'$  and  $U = \pi_{\alpha_0}^{-1}(U_{\alpha_0}) \in \mathcal{P}(M')$ , for some  $\alpha_0 \in A$ . By continuity of the scalar product of  $M_{\alpha_0}$ , there exist  $I_{a,m,\alpha_0} \in \mathcal{P}(k)$  and  $V_{a,m,\alpha_0} \in \mathcal{P}(M_{\alpha_0})$  such that

$$\begin{aligned} (a + I_{a,m,\alpha_0})(m + \pi_{\alpha_0}^{-1}(V_{a,m,\alpha_0})) &= \pi_{\alpha_0}^{-1}((a + I_{a,m,\alpha_0})(m_{\alpha_0} + V_{a,m,\alpha_0})) \\ &\subset \pi_{\alpha_0}^{-1}(am_{\alpha_0} + U_{\alpha_0}) \subset am + U. \end{aligned}$$

We now observe that if  $(M_\alpha)_{\alpha \in A}$  is a projective system in  $\mathcal{LM}_k^u$ , for the given  $U = \pi_{\alpha_0}^{-1}(U_{\alpha_0}) \in \mathcal{P}(M')$ , we may pick  $I_{a,m,\alpha_0} = I_{\alpha_0} \in \mathcal{P}(k)$  and  $V_{a,m,\alpha_0} = V_{\alpha_0} \in \mathcal{P}(M_{\alpha_0})$  independent of  $a, m$ . Then for  $I_U := I_{\alpha_0}$  and  $V_U := \pi_{\alpha_0}^{-1}(V_{\alpha_0})$ , we have

$$(a + I_U)(m + V_U) \subset am + U$$

for any  $a \in k$  and  $m \in M$ , so that the scalar product of  $M'$  is uniformly continuous. It is clear that  $(M', \tau)$  is indeed the projective limit of  $(M_\alpha)_{\alpha \in A}$  in  $\mathcal{LM}_k$ .

2. Let  $(M_\alpha)_{\alpha \in A}$  be an inductive system in  $\mathcal{LM}_k$  indexed by the preordered set  $A$ . Its inductive limit in  $\mathcal{LM}_k$  is calculated as follows. We first consider  $M' = \varinjlim_{\alpha \in A} M_\alpha^{\text{for}}$  in  $\mathcal{Mod}_k$

and let  $j_\alpha : M_\alpha^{\text{for}} \rightarrow M'$  be the natural morphisms. We then give to  $M'$  the finest  $k$ -linear topology such that all maps  $j_\alpha : M_\alpha \rightarrow M'$  are continuous. So, a basis of open  $k$ -submodules in  $M'$  consists of the  $k$ -submodules  $U$  of  $M'$  such that  $U_\alpha := j_\alpha^{-1}(U)$  is open in  $M_\alpha$ , for any  $\alpha \in A$ . Then  $\varinjlim_{\alpha \in A} M_\alpha$  is represented by the completion  $M$  of  $M'$  in that topology, equipped

with the natural morphisms  $i_\alpha : M_\alpha \rightarrow M$  deduced from the  $j_\alpha$ 's. It is clear that, for any fixed  $a \in k$ , a scalar product map

$$M \longrightarrow M, \quad x \longmapsto ax$$

is uniquely defined as the inductive limit of the maps

$$M_\alpha \longrightarrow M_\alpha, \quad x \longmapsto ax$$

and is then continuous. We must check that the scalar product of  $M$  is continuous for the product topology. So, let  $a \in k$ ,  $m \in M$ , and let  $U \in \mathcal{P}(M)$  be as before; let  $\{m_\alpha\}_{\alpha \in A}$ , with  $m_\alpha \in j_\alpha(M_\alpha)$ , be an  $A$ -net converging to  $m \in M$ . So, there exists an index  $\alpha_0 \in A$  such that  $m_\alpha \in m + U$ , for any  $\alpha \geq \alpha_0$ . We then pick  $J \in \mathcal{P}(k)$  such that  $Jm_{\alpha_0} \in j_{\alpha_0}(U_{\alpha_0}) (\subset U)$ . Then

$$(a + J)(m + U) \subset am + aU + Jm + JU \subset am + Jm_{\alpha_0} + JU \subset am + U.$$

We have to prove that  $M$  is indeed the inductive limit of the system  $\{M_\alpha\}_\alpha$ . For any  $N$  in  $\mathcal{LM}_k$ ,

(2.10.1)

$$\text{Hom}_{\mathcal{LM}_k}(M, N) = \{\varphi \in \text{Hom}_{\mathcal{Mod}_k}(\varinjlim_{\alpha \in A} M_\alpha^{\text{for}}, N) \mid \varphi \circ j_\alpha \text{ is continuous } \forall \alpha \in A\} =$$

$$\{\varphi = (\varphi_\alpha)_{\alpha \in A} \in \varprojlim_{\alpha \in A} \text{Hom}_{\mathcal{Mod}_k}(M_\alpha^{\text{for}}, N) \mid \varphi_\alpha \text{ is continuous } \forall \alpha \in A\} = \varprojlim_{\alpha \in A} \text{Hom}_{\mathcal{LM}_k}(M_\alpha, N).$$

3. Suppose  $(M_\alpha)_{\alpha \in A}$  is an inductive system in  $\mathcal{LM}_k^u$  indexed by the preordered set  $A$ . Then we slightly modify the discussion of 2 in that we equip  $M'$  the finest  $k$ -linear topology,

weaker than the naive canonical topology, such that all maps  $j_\alpha : M_\alpha \rightarrow M'$  are continuous. Then the proof of 2 can be adapted to the present situation.

4. Is clear by the construction.  $\square$

**Notation 2.11.** For an inductive system  $(M_\alpha)_{\alpha \in A}$  in  $\mathcal{LM}_k$  (resp. in  $\mathcal{LM}_k^u$ ), the inductive limit of  $(M_\alpha)_{\alpha \in A}$  in  $\mathcal{LM}_k$  (resp. in  $\mathcal{LM}_k^u$ ) will be denoted  $\varinjlim M_\alpha$  (resp.  $\varinjlim^u M_\alpha$ ). If  $(M_\alpha)_{\alpha \in A}$  is an inductive system in  $\mathcal{LM}_k^u$ , we have a canonical surjective morphism

$$(2.11.1) \quad \varinjlim M_\alpha \longrightarrow \varinjlim^u M_\alpha$$

in  $\mathcal{LM}_k$ .

**Remark 2.12.** If  $k$  is discrete an inductive system  $(M_\alpha)_{\alpha \in A}$  in  $\mathcal{LM}_k$  is also an inductive system in  $\mathcal{LM}_k^u$  and (2.11.1) is an isomorphism.

**Notation 2.13.** Let  $(M_\alpha)_{\alpha \in A}$  be an inductive system in  $\mathcal{LM}_k$  with transition morphisms  $j_{\alpha, \beta} : M_\alpha \rightarrow M_\beta$  for  $\alpha \leq \beta$ . For any  $\alpha \in A$  let  $\mathcal{P}(M_\alpha)$  denote the set of open  $k$ -submodules of  $M_\alpha$ . Then a *coherent system of open  $k$ -submodules of  $(M_\alpha)_{\alpha \in A}$*  is a system  $\mathcal{P} := (P_\alpha)_{\alpha \in A}$  such that for any  $\alpha \leq \beta$  in  $A$ ,  $j_{\alpha, \beta}^{-1}(P_\beta) = P_\alpha$ . The set  $\mathcal{C}((M_\alpha)_{\alpha \in A})$  of coherent systems of open  $k$ -submodules of  $(M_\alpha)_{\alpha \in A}$  forms a filter of  $k$ -submodules of  $\prod_{\alpha \in A} M_\alpha^{\text{for}}$ .

**Lemma 2.14.** We use the notation of (2.13).

1. Let  $(M_\alpha)_{\alpha \in A}$  be an inductive system in  $\mathcal{LM}_k$ . Then

$$(2.14.1) \quad \varinjlim M_\alpha = \varprojlim_{\mathcal{P} \in \mathcal{C}((M_\alpha)_{\alpha \in A})} M_\alpha / P_\alpha .$$

2. Let  $(M_\alpha)_{\alpha \in A}$  is an inductive system in  $\mathcal{LM}_k^u$ . Then

$$(2.14.2) \quad \varinjlim^u M_\alpha = \varprojlim_{I \in \mathcal{P}(k)} \varinjlim_{\alpha \in A} M_\alpha / \overline{IM_\alpha} ,$$

in which

$$\varinjlim_{\alpha \in A} M_\alpha / \overline{IM_\alpha} = \varinjlim_{\alpha \in A}^u M_\alpha / \overline{IM_\alpha}$$

since  $k/I$  is discrete.

*Proof.* Clear.  $\square$

**Example 2.15.** Consider the inductive system

$$(\mathbb{Z}_p, p) := \mathbb{Z}_p \xrightarrow{p} \mathbb{Z}_p \xrightarrow{p} \mathbb{Z}_p \xrightarrow{p} \dots ,$$

where  $p : \mathbb{Z}_p \rightarrow \mathbb{Z}_p$  is multiplication by  $p$ . So,  $(\mathbb{Z}_p, p)$  is an inductive system in  $\mathcal{LM}_k$ , for  $k = \mathbb{Z}_p$ . Then

$$\varinjlim (\mathbb{Z}_p, p) = \mathbb{Q}_p \text{ while } \varinjlim^u (\mathbb{Z}_p, p) = (0) .$$

**Remark 2.16.** We conclude from Proposition 2.9 that the functor

$$\begin{aligned} (-)^{\text{can}} : \mathcal{Mod}_k &\longrightarrow \mathcal{LM}_k^u \\ M &\longmapsto M^{\text{can}} \end{aligned}$$

commutes with inductive limits, that is, for any inductive system  $(M_\alpha)_{\alpha \in A}$  in  $\mathcal{Mod}_k$ ,

$$(2.16.1) \quad (\varinjlim M_\alpha)^{\text{can}} = \varinjlim^u M_\alpha^{\text{can}} ,$$



while  $M \mapsto M^{\text{for}}$  commutes with the projective ones. Actually, for any projective system  $(M_\alpha)_{\alpha \in A}$  in  $\mathcal{LM}_k$ ,

$$(2.16.2) \quad (\varprojlim_{\alpha \in A} M_\alpha)^{\text{for}} = \varprojlim_{\alpha \in A} M_\alpha^{\text{for}} .$$

On the other hand it is clear that, if for any  $\alpha \leq \beta$  the morphism  $M_\alpha \rightarrow M_\beta$  of an inductive system  $(M_\alpha)_{\alpha \in A}$  in  $\mathcal{LM}_k$  is open, then

$$(2.16.3) \quad (\varinjlim_{\alpha \in A} M_\alpha)^{\text{for}} = \varinjlim_{\alpha \in A} M_\alpha^{\text{for}}$$

equipped with the topology for which a fundamental system of open  $k$ -submodules is given by the set of  $j_\alpha(P)$ , for  $\alpha \in A$  and for  $P$  an open  $k$ -submodule of  $M_\alpha$ .

**Lemma 2.17.** *For morphism  $f : M \rightarrow N$  in  $\mathcal{LM}_k$  we have the following description.*

1.  $\text{Ker}(f) = \text{Ker}(f^{\text{for}})$ , endowed with the subspace topology in  $M$ .

2.

$$\text{Coker}(f) = \varprojlim_{Q \in \mathcal{P}(N)} N^{\text{for}} / (Q^{\text{for}} + \text{Im}(f^{\text{for}})) ,$$

where every  $N^{\text{for}} / (Q^{\text{for}} + \text{Im}(f^{\text{for}}))$  is equipped with the discrete topology.

3. If  $f$  is open, then  $\text{Coker}(f) = \text{Coker}(f^{\text{for}})$  equipped with the discrete topology.

**Remark 2.18.** The categories  $\mathcal{LM}_k$  and  $\mathcal{LM}_k^u$  are not in general abelian. In particular, for a morphism  $f : M \rightarrow N$  in any of the previous categories, we have a canonical mono/epi-morphism  $\bar{f} : \text{Coim}(f) \rightarrow \text{Im}(f)$  which permits to regard  $\text{Coim}(f)$  as a dense  $k$ -submodule of  $\text{Im}(f)$ , whose topology is finer than the subspace topology induced by the topology of  $\text{Im}(f)$ , i.e. by the one of  $N$ . When  $\bar{f}$  is an isomorphism we say that  $f$  is *strict*.

**Definition 2.19.** A sub-object  $N$  of some object  $M$  in  $\mathcal{LM}_k^u$  or in  $\mathcal{LM}_k$  is a sub-object  $N^{\text{for}}$  of  $M^{\text{for}}$  which is closed in  $M$  and is equipped with the subspace topology. The morphism  $i_N : N \rightarrow M$  will be called the *embedding of the sub-object  $N$  of  $M$* . If  $i_N$  is a continuous open injection, then  $N$  is an *open sub-object of  $M$*  and  $i_N$  will be called the *open embedding of  $N$  in  $M$* .

**Example 2.20.** Let  $\mathcal{C}(\mathbb{Q}_p, k)$  (resp.  $\mathcal{C}_{\text{unif}}(\mathbb{Q}_p, k)$ ) be the  $k$ -module of continuous (resp. uniformly continuous) functions  $\mathbb{Q}_p \rightarrow k$ , equipped with the topology of simple (resp. uniform) convergence. Then the natural morphism  $\mathcal{C}_{\text{unif}}(\mathbb{Q}_p, k) \rightarrow \mathcal{C}(\mathbb{Q}_p, k)$  in  $\mathcal{LM}_k$  is injective and has dense image, so is not the embedding of a sub-object of  $\mathcal{C}(\mathbb{Q}_p, k)$  in our sense.

**Definition 2.21.** If  $i_N : N \rightarrow M$  is a sub-object of  $M$  in  $\mathcal{LM}_k$  (resp.  $\mathcal{LM}_k^u$ ) we set

$$M/N = \text{Coker}(i_N)$$

in the category  $\mathcal{LM}_k$  (resp.  $\mathcal{LM}_k^u$ ). So,

$$M/N = \varprojlim_{Q \in \mathcal{P}(M)} M^{\text{for}} / (Q^{\text{for}} + N^{\text{for}}) ,$$

where every  $M^{\text{for}} / (Q^{\text{for}} + N^{\text{for}})$  is equipped with the discrete topology. An exact sequence is any sequence isomorphic to a sequence of the form

$$0 \longrightarrow N \xrightarrow{i_N} M \longrightarrow M/N \longrightarrow 0 .$$

**Remark 2.22.** The embedding  $i_N : N \hookrightarrow M$  of a sub-object is the same as a strict injection, since the latter is necessarily closed. Then  $N$  is an open sub-object of  $M$  if and only if the morphism  $i_N$  is an open injection.

**Lemma 2.23.** *Any open map is strict.*

*Proof.* If  $f : M \rightarrow N$  is open, then  $\text{Im}(f) = \text{Im}(f^{\text{for}})$  is an open sub-object of  $N$ , equipped with the subspace topology. Moreover, any open submodule of  $\text{Coim}(f)$  is of the form  $\overline{P} := P/\text{Ker}(f)$  where  $P \in \mathcal{P}(M)$  contains  $\text{Ker}(f)$ . Then  $\overline{f}(\overline{P}) = f(P)$ , an open subobject of  $\text{Im}(f)$ . So,  $\overline{f}$  is an isomorphism.  $\square$

**Remark 2.24.** Let  $M$  be an object of  $\mathcal{LM}_k$  whose topology is discrete. Then,

$$(2.24.1) \quad M = \bigcup_{I \in \mathcal{P}(k)} M_{[I]}$$

where

$$M_{[I]} := \{m \in M \mid am = 0, \forall a \in I\}.$$

Notice that  $M_{[I]}$ , but not  $M$  in general, is an object of  $\mathcal{LM}_k^u$ . More precisely we have:

*An object of  $\mathcal{LM}_k$  which carries the discrete topology is uniform if and only if it is an object of  $\mathcal{LM}_{k/I}$ , for some open ideal  $I$  of  $k$ .*

Notice that by Remark 2.16 formula (2.24.1) can also be written as

$$(2.24.2) \quad M = \varinjlim_{I \in \mathcal{P}(k)} M_{[I]}$$

the colimit in  $\mathcal{LM}_k$  of a filtered inductive system of discrete objects of  $\mathcal{LM}_k^u$  and injections.

**Lemma 2.25.**

1. Any object  $M$  in  $\mathcal{LM}_k$  is a projective limit of a filtered projective system of discrete  $k$ -modules and surjections

$$(2.25.1) \quad M = \varprojlim_{P \in \mathcal{P}(M)} M/P.$$

More precisely,

$$(2.25.2) \quad M = \varprojlim_{P \in \mathcal{P}(M)} \varinjlim_{I \in \mathcal{P}(k)} (M/P)_{[I]},$$

where any  $(M/P)_{[I]}$  is a discrete  $k/I$ -module.

2. Any object  $M$  in  $\mathcal{LM}_k^u$  is a projective limit of a filtered projective system of discrete uniform  $k$ -modules and surjections

$$(2.25.3) \quad M = \varprojlim_P M/P,$$

where  $P$  runs over a fundamental system of open  $k$ -submodules. Equivalently, any  $M/P$  in (2.25.3) is a discrete  $k/I$ -module, for some  $I = I_P \in \mathcal{P}(k)$ .

3. Let  $M, N$  be objects of  $\mathcal{LM}_k^u$  and let  $\mathcal{P}(M), \mathcal{P}(N)$  be fundamental systems of open  $k$ -submodules in  $M$  and  $N$ , respectively. Then

$$(2.25.4) \quad \text{Hom}_{\mathcal{LM}_k^u}(M, N) = \varprojlim_{Q \in \mathcal{P}(N)} \varinjlim_{P \in \mathcal{P}(M)} \text{Hom}_k(M/P, N/Q)$$

as a  $k^{\text{for}}$ -module. Notice that both  $M/P$  and  $N/Q$  are  $k/I$ -modules for some  $I \in \mathcal{P}(k)$ , so that  $\text{Hom}_k(M/P, N/Q) = \text{Hom}_{k/I}(M/P, N/Q)$  for any such  $I$ .

4. Let  $M, N$  be objects of  $\mathcal{LM}_k$  and let  $\mathcal{P}(M), \mathcal{P}(N)$  be fundamental systems of open  $k$ -submodules in  $M$  and  $N$ , respectively. Then

$$(2.25.5) \quad \text{Hom}_{\mathcal{LM}_k}(M, N) = \varprojlim_{Q \in \mathcal{P}(N)} \varinjlim_{P \in \mathcal{P}(M)} \varprojlim_{I \in \mathcal{P}(k)} \text{Hom}_{k/I}((M/P)_{[I]}, (N/Q)_{[I]})$$

as a  $k^{\text{for}}$ -module.

*Proof.*

The statement in 1 means that for an object  $M$  in  $\mathcal{LM}_k$ , the underlying  $k^{\text{for}}$ -module  $M^{\text{for}}$  is a projective limit in  $\text{Mod}_k$  of a filtered projective system of  $k^{\text{for}}$ -modules

$$(2.25.6) \quad M^{\text{for}} = \varprojlim_{P \in \mathcal{P}(M)} M^{\text{for}}/P^{\text{for}}.$$

The topology of  $M$  is the weak topology of the projections  $\pi_P : M^{\text{for}} \rightarrow M^{\text{for}}/P^{\text{for}}$ , where the target is equipped with the discrete topology. All this is clear. The remaining part of 1 follows from Remark 2.24.

2 follows from the definition and from Remark 2.24.

3, 4 follow from 2 and 1, respectively.  $\square$

**Lemma 2.26.** *Let  $f : N \rightarrow M$  be a morphism in  $\mathcal{LM}_k$  and let (2.25.1) be the representation of  $M$  as a limit of discrete objects where  $\pi_P : M \rightarrow M/P$  and  $\pi_{P,Q} : M/P \rightarrow M/Q$  are the canonical projections, for any  $P \subset Q$  in  $\mathcal{P}(M)$ . For any  $P \in \mathcal{P}(M)$  let  $\overline{N}_P$  be the set-theoretic image of the natural morphism  $f|_P : N \rightarrow M/P =: \overline{M}_P$ . Then*

1.  *$f$  is the embedding of a closed sub-object if and only if the morphism*

$$(2.26.1) \quad F : N \longrightarrow \varprojlim_{P \in \mathcal{P}(M)} \overline{N}_P$$

*deduced from  $\{f|_P\}_P$  by the universal property of the projective limit is an isomorphism.*

2. *Let  $N$  be a sub-object of  $M$  as in (2.26.1). Then the quotient  $M/N$  is*

$$(2.26.2) \quad M/N \xrightarrow{\sim} \varprojlim_{P \in \mathcal{P}(M)} \overline{M}_P / \overline{N}_P,$$

3. *Assume  $f$  is a closed embedding. Then  $f$  is open if and only if there exists  $P_0 \in \mathcal{P}(M)$  such that, for any  $P \subset Q \subset P_0$  in  $\mathcal{P}(M)$ , the map  $\pi_{P,Q} : M/P \rightarrow M/Q$  induces a surjection  $\overline{N}_P \rightarrow \overline{N}_Q$ .*

*Proof.* All the assertions are clear.  $\square$

**Proposition 2.27.** *An object  $M$  of  $\mathcal{LM}_k$  is canonical if and only if it is a projective limit of a projective system  $\{\overline{M}_I\}_{I \in \mathcal{P}(k)}$ , indexed by  $I \in \mathcal{P}(k)$ , where each  $\overline{M}_I$  is a discrete  $k/I$ -module and, for any  $I \subset J$  in  $\mathcal{P}(k)$ , the morphism  $\overline{M}_I \rightarrow \overline{M}_J$  is a surjection. So,*

$$(2.27.1) \quad M = \varprojlim_{I \in \mathcal{P}(k)} \overline{M}_I.$$

*Proof.* Clear.  $\square$

**Remark 2.28.** We observe that an open sub-object  $N$  of a canonical module  $M$  carries the canonical topology. In fact,  $N$  correspond to a projective system  $\overline{N}_I$  of  $k/I$ -submodules of  $\overline{M}_I = M/\overline{IM}$  such that the morphisms  $\overline{N}_J \rightarrow \overline{N}_I$ , for  $J \subset I$  in  $\mathcal{P}(k)$ , are surjective for sufficiently small  $I$ . So  $\{\overline{N}_I\}_{I \in \mathcal{P}(k)}$  satisfies the condition in Definition 2.27.

So, if  $M$  is canonical, a basis of open  $k$ -submodules of  $M$  consists of the closures  $\overline{IM} = \pi_I^{-1}(\overline{M_I})$  in  $M$  of the submodules  $IM$ , for  $I$  in a basis of open ideals of  $k$ , where  $\pi_I : M \rightarrow \overline{M_I}$  denotes the canonical projection. For any  $M$  in  $\mathcal{LM}_k$ , the *canonical topology* of  $M$  is the topology for which  $\{\overline{IM}\}_{I \in \mathcal{P}(k)}$  is a fundamental system of open neighborhoods of 0. So,  $M$  is separated in its canonical topology if and only if  $\bigcap_{I \in \mathcal{P}(k)} \overline{IM} = (0)$ .

An object of  $\mathcal{LM}_k$  is not necessarily separated in its canonical topology as one sees in the case of the object  $\mathbb{Q}_p$  of  $\mathcal{LM}_{\mathbb{Z}_p}$ . The  $p$ -adic topology of  $\mathbb{Q}_p$  is not  $\mathbb{Z}_p$ -canonical, since for any open ideal  $I = p^N \mathbb{Z}_p$  of  $\mathbb{Z}_p$ ,  $I\mathbb{Q}_p = \mathbb{Q}_p$ .

For any  $M$  in  $\mathcal{LM}_k$  there is a natural exact sequence

$$(2.28.1) \quad 0 \longrightarrow \bigcap_{I \in \mathcal{P}(k)} \overline{IM} \longrightarrow M \xrightarrow{\gamma_M} \varprojlim_{I \in \mathcal{P}(k)} M/\overline{IM} \longrightarrow 0.$$

**Lemma 2.29.** *Let  $M$  be an object of  $\mathcal{LM}_k$ , and let (2.28.1) be the corresponding exact sequence. Then*

1.  *$M$  is uniform if and only if its topology is weaker than the canonical topology.*
2.  *$M$  is uniform if and only if  $\bigcap_{I \in \mathcal{P}(k)} \overline{IM} = (0)$ .*
3.  *$\varprojlim_{I \in \mathcal{P}(k)} M/\overline{IM}$  is the maximal uniform quotient of  $M$ .*

*Proof.* The first part of the statement follows from Lemma 2.5. The other parts are clear.  $\square$

**Remark 2.30.** It follows from Remark 2.2 that if  $k$  is a domain and satisfies **OP** then, on any open ideal  $J$  of  $k$ , the subspace topology of  $J$  in  $k$ , the canonical topology of  $J$  and the naive canonical topology of  $J$  all coincide.

**Proposition 2.31.** *Assume the object  $M$  of  $\mathcal{LM}_k$  is equipped with the canonical topology and  $k$  satisfies **OP**. Assume moreover that  $M$  satisfies*

**(TFM)** For any  $I \in \mathcal{P}(k)$ ,  $M/\overline{IM}$  is a torsion-free  $k/I$ -module.

*Then, condition **OPM** holds for the  $k$ -linearly topologized  $k$ -module  $M$ .*

*Proof.* We may assume that  $M = \widehat{N}$  as in (2.27.1), where  $N$  is a  $k$ -module equipped with the naive canonical topology. By condition **OP**,  $ak$  is open in  $k$  and  $aN$  is open in  $N$ . We have to show that  $aM$  is open in  $M$ . Equivalently, we show that  $aM \supset \overline{aN}$  in  $M$ . Let  $ax_\alpha$ , for  $x_\alpha \in N$  for any  $\alpha \in A$ , be a net converging to  $y \in M$ . We need to show that  $\{x_\alpha\}_{\alpha \in A}$  converges in  $M$ . So, it suffices to show that if the net  $\{ax_\alpha\}_{\alpha \in A}$  is Cauchy in  $N$ , the net  $\{x_\alpha\}_{\alpha \in A}$  is Cauchy in  $N$ , as well. This follows from condition **TFM**. In fact, for any  $I \in \mathcal{P}(k)$ ,  $M/\overline{IM} \cong N/IN$  is a torsion-free  $k/I$ -module. Then, let  $I \in \mathcal{P}(k)$  be such that  $I \subsetneq ak$ , and let  $\alpha_0 \in A$  be such that  $a(x_\alpha - x_\beta) \in IN$  for all  $\alpha, \beta \geq \alpha_0$ . By condition **TFM**,  $x_\alpha - x_\beta \in IN$  for all  $\alpha, \beta \geq \alpha_0$ .  $\square$

**Lemma 2.32.** *Let  $i_N : N \rightarrow M$  be an open embedding in  $\mathcal{LM}_k$  with  $M$  canonical. Then  $N$  is canonical.*

*Proof.* This follows from the description of open sub-objects of  $M$  in 3 of Lemma 2.26 and from the Definition 2.27.1 of canonical modules.  $\square$

**Definition 2.33.** *Let  $M$  be an object of  $\mathcal{LM}_k$ . A formal series  $\sum_{\alpha \in A} m_\alpha$  of elements  $m_\alpha \in M$ , indexed by any set  $A$ , converges unconditionally to  $m \in M$  if the net  $F \mapsto \sigma_F := \sum_{\alpha \in F} m_\alpha$ , for  $F$  a finite subset of  $A$ , converges to  $m$ . We then say that  $m$  is the sum of the  $A$ -series  $\sum_{\alpha \in F} m_\alpha$  and write  $m = \sum_{\alpha \in A} m_\alpha$ .*

### 3 Complete tensor products

The category  $\mathcal{LM}_k$  and its full subcategory  $\mathcal{LM}_k^u$  admit various natural notions of complete tensor products. We are interested in two of them.

**Definition 3.1.** For 3 objects  $M, N, P$  of  $\mathcal{LM}_k$ , we denote by  $\text{Bil}_k^c(M \times N, P)$  (resp.  $\text{Bil}_k^u(M \times N, P)$ ) the  $k$ -module of  $k$ -bilinear functions  $f : M \times N \rightarrow P$  which are continuous for the product topology of  $M \times N$  (resp. uniformly continuous for the product uniformity of  $M \times N$ ). We denote by  $M \widehat{\otimes}_k^c N$  (resp.  $M \widehat{\otimes}_k^u N$ ) the object of  $\mathcal{LM}_k$ , if it exists, which represents the functor  $\mathcal{LM}_k \rightarrow \text{Mod}_k \quad P \mapsto \text{Bil}_k^c(M \times N, P)$  (resp.  $P \mapsto \text{Bil}_k^u(M \times N, P)$ ).

**Proposition 3.2.** Let  $M = \varinjlim_{\alpha} M_{\alpha}$  and  $N = \varinjlim_{\beta} N_{\beta}$  be objects of  $\mathcal{LM}_k$ , for filtered inductive systems  $\{M_{\alpha}\}_{\alpha}$  and  $\{N_{\beta}\}_{\beta}$  in  $\mathcal{LM}_k$ . Then, for any  $X$  in  $\mathcal{LM}_k$ , there is a canonical isomorphism in  $\text{Mod}_k$

$$(3.2.1) \quad \begin{aligned} \text{Bil}_k^c(M \times N, X) &\xrightarrow{\sim} \varprojlim_{\alpha, \beta} \text{Bil}_k^c(M_{\alpha} \times N_{\beta}, X) \\ \varphi &\longmapsto (\varphi \circ j_{\alpha, \beta})_{\alpha, \beta} \end{aligned}$$

where  $j_{\alpha, \beta} : M_{\alpha} \times N_{\beta} \rightarrow M \times N$  is the canonical morphism.

*Proof.* The  $k$ -linear map of (3.2.1) clearly exists and is injective. We have to show surjectivity. So, let  $(\varphi_{\alpha, \beta})_{\alpha, \beta}$  be a compatible system in  $\varprojlim_{\alpha, \beta} \text{Bil}_k^c(M_{\alpha} \times N_{\beta}, X)$ . For any

$(m, n) \in M \times N = \varinjlim_{\alpha, \beta} M_{\alpha} \times N_{\beta}$ , let  $(j_{\alpha, \beta}(m_{\alpha}, n_{\beta}))_{\alpha, \beta}$  be a net indexed by  $A \times B$  con-

verging to  $(m, n)$ , with  $(m_{\alpha}, n_{\beta}) \in M_{\alpha} \times N_{\beta}$ . Then the net  $(\varphi_{\alpha, \beta}(m_{\alpha}, n_{\beta}))_{\alpha, \beta}$  is a Cauchy net in  $X$ . It suffices to check that for  $(m, n) = (0, 0)$  the previous net converges to  $0 \in X$ . Now, for any  $U \in \mathcal{P}(X)$  and any  $(\alpha, \beta)$ , there are  $V_{\alpha} \in \mathcal{P}(M_{\alpha})$  and  $W_{\beta} \in \mathcal{P}(N_{\beta})$  such that  $\varphi_{\alpha, \beta}(V_{\alpha} \times W_{\beta}) \subset U$ . By definition of inductive limits in  $\mathcal{LM}_k$ , there exists  $V \in \mathcal{P}(M)$  (resp.  $W \in \mathcal{P}(N)$ ) such that, for any  $(\alpha, \beta) \in A \times B$ ,  $j_{\alpha, \beta}^{-1}(V \times W) \subset V_{\alpha} \times W_{\beta}$ .

Since  $(j_{\alpha, \beta}(m_{\alpha}, n_{\beta}))_{\alpha, \beta}$  converges to  $(0, 0) \in M \times N$ , we may then assume that  $(m_{\alpha}, n_{\beta}) \in V_{\alpha} \times W_{\beta}$  for any  $\alpha, \beta$ . It then follows that  $\varphi_{\alpha, \beta}(m_{\alpha} \times n_{\beta}) \in U$ , for any  $\alpha, \beta$ .

Coming back to the case of any  $(m, n) \in M \times N$ , we define  $\varphi(m, n)$  as the limit of the net  $(\varphi_{\alpha, \beta}(m_{\alpha}, n_{\beta}))_{\alpha, \beta}$ . The definition is good since it is for  $(m, n) = (0, 0)$ . It is clear that

$$\varphi \circ j_{\alpha, \beta} = \varphi_{\alpha, \beta} \quad \forall (\alpha, \beta) \in A \times B.$$

□

**Lemma 3.3.** Let  $M$  and  $N$  be objects of  $\mathcal{LM}_k$ .

1.  $M \widehat{\otimes}_k^c N$  exists in  $\mathcal{LM}_k$ . More precisely,  $M \widehat{\otimes}_k^c N$  is the completion of  $M \otimes_k N$  in the  $k$ -linear topology with a fundamental set of open  $k$ -submodules given by the images  $\text{Im}(P \otimes_k Q)$  in  $M \otimes_k N$ , for  $P$  (resp.  $Q$ ) varying in the set of open submodules of  $M$  (resp.  $N$ ). So,

$$(3.3.1) \quad M \widehat{\otimes}_k^c N = \varprojlim_{P, Q} (M \otimes_k N) / \text{Im}(P \otimes_k Q)$$

for  $P$  (resp.  $Q$ ) as before, where all the terms of the projective systems carry the discrete topology.

2.  $M \widehat{\otimes}_k^u N$  exists in  $\mathcal{LM}_k$ . More precisely  $M \widehat{\otimes}_k^u N$  is the completion of  $M \otimes_k N$  in the  $k$ -linear topology with a fundamental set of open  $k$ -submodules given by the images

$\text{Im}(P \otimes_k N + M \otimes_k Q)$  in  $M \otimes_k N$ , for  $P$  (resp.  $Q$ ) varying in the set of open submodules of  $M$  (resp.  $N$ ). We have

$$(3.3.2) \quad M \widehat{\otimes}_k^u N = \varprojlim_{P, Q} (M \otimes_k N) / \text{Im}(P \otimes_k N + M \otimes_k Q) = \varprojlim_{P, Q} M / P \otimes_k N / Q ,$$

for  $P$  (resp.  $Q$ ) as before, where all the terms of the projective systems carry the discrete topology and are uniform. A fundamental system of open submodules of  $M \widehat{\otimes}_k^u N$  consists of the closures in  $M \widehat{\otimes}_k^u N$  of the  $k$ -submodules  $\text{Im}(P \otimes_k N + M \otimes_k Q) \subset M \otimes_k N$ , for  $P, Q$  as before.

*Proof.* Formula (3.3.1) is proven in essentially the same way as [12, Lemma 17.1]. Namely, we let  $\mathcal{P}(M)$  (resp.  $\mathcal{P}(N)$ ) be a basis of open  $k$ -submodules of  $M$  (resp.  $N$ ). Then, for any object  $T$  of  $\mathcal{LM}_k$  with a basis of open  $k$ -submodules  $\mathcal{P}(T)$ , a continuous  $k$ -bilinear map  $\beta : M \times N \rightarrow T$  is in particular continuous at  $(0, 0) \in M \times N$ . Let  $\beta' : M \otimes_k N \rightarrow T$  be the  $k$ -linear map corresponding to  $\beta$ . For any  $R \in \mathcal{P}(T)$  there are  $P \in \mathcal{P}(M)$  and  $Q \in \mathcal{P}(N)$  such that  $\beta(P \times Q) \subset R$  and therefore  $\beta'(\text{Im}(P \otimes_k Q)) \subset R$ . We conclude that  $\beta'$  extends to a continuous  $k$ -linear map  $\gamma := \widehat{\beta}' : M \widehat{\otimes}_k^c N \rightarrow T$  such that

$$(3.3.3) \quad \gamma(m \widehat{\otimes}_k^c n) = \beta(m, n) .$$

Conversely, given the morphism  $\gamma : M \widehat{\otimes}_k^c N \rightarrow T$  in  $\mathcal{LM}_k$ , the  $k$ -bilinear map  $\beta : M \times N \rightarrow T$  defined by (3.3.3) is such that for any  $R \in \mathcal{P}(T)$  there are  $P \in \mathcal{P}(M)$  and  $Q \in \mathcal{P}(N)$  such that  $\beta(P \times Q) \subset R$ . Moreover, for any  $m \in M$  (resp.  $n \in N$ ) the  $k$ -linear map  $N \rightarrow T$  given by  $y \mapsto \beta(m, y)$  (resp.  $M \rightarrow T$  given by  $x \mapsto \beta(x, n)$ ) is continuous at 0, hence is uniformly continuous. So, for any fixed  $(m, n) \in M \times N$  we can find  $P_n \in \mathcal{P}(M)$  and  $Q_m \in \mathcal{P}(N)$  such that

$$\beta(m \times Q_m), \beta(P_n, n), \beta(P_n, Q_m) \subset R .$$

Then

$$\beta((m + P_n) \times (n + Q_m)) \subset \beta(m, n) + \beta(\{m\} \times Q_m) + \beta(P_n \times \{n\}) + \beta(P_n \times Q_m) \subset \beta(m, n) + R .$$

This proves that  $\beta$  is continuous for the product topology of  $M \times N$ .

We now pass to (3.3.2) : we prove the first equality in that formula. Let  $\beta : M \times N \rightarrow P$  be  $k$ -bilinear and uniformly continuous. Then, for any open submodule  $W$  of  $P$ , we can find an open submodule  $U$  (resp  $V$ ) of  $M$  (resp  $N$ ) such that, for any  $(x, y) \in M \times N$

$$\beta(x + U, y + V) = \beta(x, y) + \beta(x, V) + \beta(U, y) + \beta(U \times V) \subset \beta(x, y) + W .$$

This means that we must have  $\beta(U \times N + M \times V) \subset W$ . Conversely, if  $\beta : M \times N \rightarrow P$  is  $k$ -bilinear and satisfies the latter condition the same calculation read backwards shows that  $\beta$  is uniformly continuous. The second equality in (3.3.2) follows from the canonical isomorphism

$$M / P \otimes_k N / Q \xrightarrow{\sim} (M \otimes_k N) / (P \otimes_k N + M \otimes_k Q)$$

proven in [4, II, §3, n. 6, Cor. 1 of Prop. 6, p. 60].

The fact that a fundamental system of open submodules of  $M \widehat{\otimes}_k^u N$  consists of the closures of  $P \otimes_k N + M \otimes_k Q$ , for  $P, Q$  in a fundamental system of open submodules of  $M, N$ , respectively, is a general fact about completions.  $\square$

**Remark 3.4.** For any  $M$  and  $N$  in  $\mathcal{LM}_k$ , we have a canonical morphism in  $\mathcal{LM}_k$

$$\Phi : M \widehat{\otimes}_k^c N \longrightarrow M \widehat{\otimes}_k^u N$$

such that, for any  $X$  in  $\mathcal{LM}_k$ ,  $\varphi \mapsto \varphi \circ \Phi$  is the natural inclusion

$$\text{Bil}_k^u(M \times N, X) \longrightarrow \text{Bil}_k^c(M \times N, X) .$$

**Proposition 3.5.**

1. Let  $M = \varinjlim_{\alpha} M_{\alpha}$  and  $N = \varinjlim_{\beta} N_{\beta}$  be objects of  $\mathcal{LM}_k$ , for inductive systems  $\{M_{\alpha}\}_{\alpha}$  and  $\{N_{\beta}\}_{\beta}$  in  $\mathcal{LM}_k$ . Then

$$(3.5.1) \quad \varinjlim_{\alpha} M_{\alpha} \widehat{\otimes}_k^c \varinjlim_{\beta} N_{\beta} = \varinjlim_{\alpha, \beta} M_{\alpha} \widehat{\otimes}_k^c N_{\beta} .$$

2. Let  $M = \varprojlim_{\alpha} M_{\alpha}$  and  $N = \varprojlim_{\beta} N_{\beta}$  be objects of  $\mathcal{LM}_k$ , for projective systems  $\{M_{\alpha}\}_{\alpha}$  and  $\{N_{\beta}\}_{\beta}$  in  $\mathcal{LM}_k$ . Then

$$(3.5.2) \quad \varprojlim_{\alpha} M_{\alpha} \widehat{\otimes}_k^u \varprojlim_{\beta} N_{\beta} = \varprojlim_{\alpha, \beta} M_{\alpha} \widehat{\otimes}_k^u N_{\beta} .$$

3. If  $M$  and  $N$  are objects of  $\mathcal{LM}_k^u$ , then both  $M \widehat{\otimes}_k^c N$  and  $M \widehat{\otimes}_k^u N$  are uniform.

*Proof.* It suffices to prove that the functor  $\text{Bil}_k^c(M \times N, -)$  of (3.1) is represented by  $\varinjlim_{\alpha, \beta} M_{\alpha} \widehat{\otimes}_k^c N_{\beta}$ . So, we pick an object  $X$  of  $\mathcal{LM}_k$  and consider (3.2.1). The l.h.s. of that equation in fact equals  $\text{Hom}_{\mathcal{LM}_k}(M \widehat{\otimes}_k^c N, X)$  while its r.h.s. equals

$$\varprojlim_{\alpha, \beta} \text{Hom}_{\mathcal{LM}_k}(M_{\alpha} \widehat{\otimes}_k^c N_{\beta}, X) = \text{Hom}_{\mathcal{LM}_k}(\varinjlim_{\alpha, \beta} M_{\alpha} \widehat{\otimes}_k^c N_{\beta}, X) .$$

So, equation (3.5.1) follows.

To prove equation (3.5.2) we write

$$M_{\alpha} = \varprojlim_{P_{\alpha} \in \mathcal{P}(M_{\alpha})} M_{\alpha}/P_{\alpha} , \quad N_{\beta} = \varprojlim_{Q_{\beta} \in \mathcal{P}(N_{\beta})} N_{\beta}/Q_{\beta} .$$

Then

$$(3.5.3) \quad \begin{aligned} \varprojlim_{\alpha} M_{\alpha} \widehat{\otimes}_k^u \varprojlim_{\beta} N_{\beta} &= (\varprojlim_{\alpha} \varprojlim_{P_{\alpha} \in \mathcal{P}(M_{\alpha})} M_{\alpha}/P_{\alpha}) \widehat{\otimes}_k^u (\varprojlim_{\beta} \varprojlim_{Q_{\beta} \in \mathcal{P}(N_{\beta})} N_{\beta}/Q_{\beta}) = \\ &= \varprojlim_{\alpha} \varprojlim_{P_{\alpha} \in \mathcal{P}(M_{\alpha})} \varprojlim_{\beta} \varprojlim_{Q_{\beta} \in \mathcal{P}(N_{\beta})} (M_{\alpha}/P_{\alpha} \otimes_k N_{\beta}/Q_{\beta}) = \varprojlim_{\alpha, \beta} M_{\alpha} \widehat{\otimes}_k^u N_{\beta} . \end{aligned}$$

Point 3 is immediate. □

## 4 Pseudoconvexity

**Definition 4.1.** An object  $M$  of  $\mathcal{LM}_k$  is pseudobanach or a pseudobanach  $k$ -module if there exists a family  $\mathcal{G}$  of open sub-objects of  $M$  satisfying the following conditions

1. Any  $P \in \mathcal{G}$  is equipped with the canonical topology.
2. For any  $P \in \mathcal{G}$  and any open ideal  $I$  of  $k$ , let  $\overline{IP}$  be the closure of  $IP$  in  $P$ . Then, the discrete  $k/I$ -module  $P/\overline{IP}$  is flat.
3.  $M$  is the union of its open submodules  $P$ , for  $P \in \mathcal{G}$ .

We call such a  $\mathcal{G}$  a  $k$ -gauge (or simply a gauge if there is no risk of confusion) of  $M$ . The full subcategory of  $\mathcal{LM}_k$  consisting of pseudobanach objects will be denoted by  $\mathcal{PB}_k$ .

**Remark 4.2.** For any  $M$  in  $\mathcal{PB}_k$  and any gauge  $\mathcal{G}$  for  $M$ , we have

$$M = \varprojlim_{P, I} M / \overline{IP}$$

where  $P \in \mathcal{G}$  and  $I$  describes the open ideals of  $k$ . Notice that the closure  $\overline{IP}$  of  $IP$  in  $P$  is open in  $M$ . So, while it is not required that  $\mathcal{G}$  should be a fundamental system of open  $k$ -submodules of  $M$ , this is certainly the case for  $\{\overline{IP}\}_{P, I}$  for  $P$  and  $I$  as before.

**Remark 4.3.** Notation as in Remark 4.2. For any  $P, Q \in \mathcal{G}$  we have that  $P \cap Q$  is an open  $k$ -submodule of both  $P$  and  $Q$ . It follows that there are open ideals  $I, J$  of  $k$  such that  $IP \subset Q$  and  $JQ \subset P$ . So, for any  $P \in \mathcal{G}$ ,  $\{\overline{IP}\}_I$ , for  $I$  an open ideal of  $k$ , is a fundamental system of open  $k$ -submodules of  $M$ .

**Remark 4.4.** Assume  $k$  satisfies condition **OP**. Then follows from Remark 2.30 that any open ideal  $J$  of  $k$  is a pseudobanach  $k$ -module with gauge the set  $\mathcal{P}(J)$  of open ideals of  $k$  contained in  $J$ .

**Remark 4.5.** If  $k$  has the discrete topology, a pseudobanach  $k$ -module  $M$  is simply a flat  $k$ -module equipped with the discrete topology. A gauge for such an  $M$  is  $\mathcal{G} = \{M\}$ . So in this case  $\mathcal{PB}_k$  is the full subcategory of  $\mathcal{LM}_k$  consisting of flat  $k$ -modules equipped with the discrete topology and  $-\widehat{\otimes}_k^c - = - \otimes_k -$ .

**Remark 4.6.** It follows from condition 2 of Definition 4.1 that, for any pseudobanach  $k$ -module  $M$ , condition **(TFM)** of Proposition 2.31 is satisfied for any  $P$  in a gauge  $\mathcal{G}$  for  $M$ . We then conclude from that proposition that, if  $k$  satisfies condition **OP**, then, for any pseudobanach  $k$ -module  $M$  and for any  $a \in k - \{0\}$ , the map

$$M \longrightarrow M, \quad x \longmapsto ax$$

is open.

**Remark 4.7.** Let  $K$  be a non-archimedean field and let  $k = K^\circ$ ; then  $K$  is a pseudobanach object of  $\mathcal{LR}_{K^\circ}$  for the gauge  $\{aK^\circ\}_{a \in K^\times}$ . Let  $M$  be an object of  $\mathcal{LM}_k$  such that for any  $a \in k - \{0\}$ , the map  $m \mapsto am$  is a bijection. Then the scalar product  $k \times M \rightarrow M$  extends uniquely to a structure of  $K$ -vector space  $K \times M \rightarrow M$ . By Remark 4.6 this is in fact a structure of topological  $K$ -vector space. In this situation we will simply say that the object  $M$  of  $\mathcal{LM}_k$  is a *topological  $K$ -vector space*.

**Definition 4.8.** Let  $K$  be a non-archimedean field and let  $k = K^\circ$ . A pseudobanach space over  $K$  is any object  $M$  of  $\mathcal{PB}_k$  which is a  $K$ -vector space. We view the category  $\mathcal{Ban}_K$  of pseudobanach spaces over  $K$  as a full subcategory of  $\mathcal{LM}_k$ .

**Corollary 4.9.** For a non-trivially valued  $K$  the category  $\mathcal{Ban}_K$  is equivalent to the category of  $K$ -Banach spaces and continuous maps of [12]. For  $K$  trivially valued the category  $\mathcal{Ban}_K$  is equivalent to  $\mathcal{Mod}_K$ .

## 5 Topological rings

**Definition 5.1.** A (topological)  $k$ -ring  $A$  is a  $k$ -linearly topologized topological  $k$ -module  $A$  such that the product map

$$\mu_A : A \times A \longrightarrow A, \quad (x, y) \longmapsto xy$$



makes  $A$  into a  $k$ -algebra (commutative with 1) and is continuous for the product topology of  $A \times A$ . We let  $\mathcal{LR}_k$  be the category of complete topological  $k$ -rings and continuous  $k$ -algebra homomorphisms. We denote by  $\mathcal{LR}_k^u$  the full subcategory of  $\mathcal{LR}_k$  consisting of the objects  $A$  such that the scalar product

$$(\mu_A)|_{k \times A} : k \times A \longrightarrow A, \quad (\lambda, y) \longmapsto \lambda y$$

is uniformly continuous. We define  $\mathcal{RR}_k$  as the full subcategory of  $\mathcal{LR}_k$  (and of  $\mathcal{LR}_k^u$ ) consisting of complete linearly topologized  $k$ -rings.

If  $A$  is an object of  $\mathcal{RR}_k$  then the product map is in fact uniformly continuous for the product uniformity of  $A \times A$  but is not necessarily open.

**Lemma 5.2.** *The categories  $\mathcal{LR}_k$ ,  $\mathcal{LR}_k^u$  and  $\mathcal{RR}_k$  admit both limits and colimits. The  $\mathcal{RR}_k$ -limit of a projective system of elements of  $\mathcal{RR}_k$  coincides with its  $\mathcal{LR}_k^u$ -limit and with its  $\mathcal{LR}_k$ -limit. The  $\mathcal{LR}_k^u$ -limit of a projective system of elements of  $\mathcal{LR}_k^u$  coincides with its  $\mathcal{LR}_k$ -limit.*

*Proof.* The case of limits in  $\mathcal{RR}_k$  follows from general nonsense. Namely, let  $(R_\alpha)_{\alpha \in A}$  be a projective system in  $\mathcal{RR}_k$ . We then equip the projective limit  $R := \varprojlim_{\alpha} R_\alpha$  of  $(R_\alpha)_{\alpha \in A}$  in  $\mathcal{LM}_k^u$  with a product map as follows. For any  $\alpha \in A$ , the product map  $\mu_{R_\alpha} : R_\alpha \times R_\alpha \rightarrow R_\alpha$  factors in this case through a morphism  $\mu_\alpha : R_\alpha \widehat{\otimes}_k^u R_\alpha \rightarrow R_\alpha$ . We then have a projective system of morphisms in  $\mathcal{LM}_k^u$

$$R \widehat{\otimes}_k^u R = (\varprojlim_{\alpha} R_\alpha) \widehat{\otimes}_k^u (\varprojlim_{\alpha} R_\alpha) = \varprojlim_{\alpha} (R_\alpha \widehat{\otimes}_k^u R_\alpha) \xrightarrow{\pi_\alpha \widehat{\otimes}_k^u \pi_\alpha} R_\alpha \widehat{\otimes}_k^u R_\alpha \xrightarrow{\mu_\alpha} R_\alpha \longrightarrow R_\beta,$$

for all  $\alpha \geq \beta$ , from which we obtain

$$\mu : R \widehat{\otimes}_k^u R \longrightarrow \varprojlim_{\beta} R_\beta$$

and finally the product map

$$\mu_R : R \times R \longrightarrow R.$$

The case of limits in  $\mathcal{LR}_k$  and  $\mathcal{LR}_k^u$  is similar to the one of  $\mathcal{LM}_k$  and  $\mathcal{LM}_k^u$  discussed in Lemma 2.25 and will be omitted.

We now prove the existence of the colimit in  $\mathcal{LR}_k$  of the inductive system  $(R_\alpha)_{\alpha \in A}$  in  $\mathcal{LR}_k$ . For any  $\alpha \in A$ , the product map  $\mu_{R_\alpha} : R_\alpha \times R_\alpha \rightarrow R_\alpha$  factors through a morphism  $\mu_\alpha : R_\alpha \widehat{\otimes}_k^c R_\alpha \rightarrow R_\alpha$ . We then equip the colimit  $R := \varinjlim_{\alpha} R_\alpha$  of the system  $(R_\alpha)_{\alpha \in A}$  in  $\mathcal{LM}_k$  with the product map obtained as follows. From the system of morphisms

$$\mu_\alpha : R_\alpha \widehat{\otimes}_k^c R_\alpha \longrightarrow R_\alpha \xrightarrow{j_\alpha} R$$

and

$$R \widehat{\otimes}_k^c R = (\varinjlim_{\alpha} R_\alpha) \widehat{\otimes}_k^c (\varinjlim_{\alpha} R_\alpha) \longrightarrow R_\alpha \widehat{\otimes}_k^c R_\alpha$$

we obtain

$$\mu := \varinjlim_{\alpha} \mu_\alpha : R \widehat{\otimes}_k^c R = (\varinjlim_{\alpha} R_\alpha) \widehat{\otimes}_k^c (\varinjlim_{\alpha} R_\alpha) \longrightarrow \varinjlim_{\alpha} (R_\alpha \widehat{\otimes}_k^c R_\alpha) \longrightarrow R$$

and finally the product map

$$\mu_R : R \times R \longrightarrow R.$$

Similarly, if the inductive system  $(R_\alpha)_{\alpha \in A}$  consists of objects of  $\mathcal{LR}_k^u$  (resp.  $\mathcal{RR}_k$ ), we equip the colimit  $R^u := \varinjlim_\alpha R_\alpha$  of the system  $(R_\alpha)_{\alpha \in A}$  in  $\mathcal{LM}_k^u$  with the product map obtained as follows. From the system of morphisms

$$\mu_\alpha : R_\alpha \widehat{\otimes}_k^c R_\alpha \longrightarrow R_\alpha \xrightarrow{j_\alpha} R,$$

where  $R_\alpha \widehat{\otimes}_k^c R_\alpha$  is uniform by comma  $\mathcal{B}$  of Proposition 3.5, and

$$R^u \widehat{\otimes}_k^c R^u = (\varinjlim_\alpha R_\alpha) \widehat{\otimes}_k^c (\varinjlim_\alpha R_\alpha) \longrightarrow R_\alpha \widehat{\otimes}_k^c R_\alpha$$

we obtain

$$\mu_{R^u} := \varinjlim_\alpha \mu_\alpha : R^u \widehat{\otimes}_k^c R^u = (\varinjlim_\alpha R_\alpha) \widehat{\otimes}_k^c (\varinjlim_\alpha R_\alpha) \longrightarrow \varinjlim_\alpha (R_\alpha \widehat{\otimes}_k^c R_\alpha) \longrightarrow R^u$$

and finally the continuous product map

$$\mu_{R^u} : R^u \times R^u \longrightarrow R^u.$$

□

**Remark 5.3.** Although not logically necessary, we prefer to give an explicit description of the product map of  $R$ . As in Lemma 2.25, we first consider  $R' = \varinjlim_{\alpha \in A} R_\alpha^{\text{for}}$  in  $\mathcal{Rings}$  and let  $j_\alpha : R_\alpha^{\text{for}} \rightarrow R'$  be the natural morphisms. We then give to  $R'$  the finest  $k$ -linear topology such that all maps  $j_\alpha : R_\alpha \rightarrow R'$  are continuous. So, a basis of open  $k$ -submodules in  $R'$  consists of the  $k$ -submodules  $U$  of  $R'$  such that  $j_\alpha^{-1}(U)$  is an open  $k$ -submodule  $J_\alpha$  of  $R_\alpha$ , for any  $\alpha \in A$ . Then  $\varinjlim_{\alpha \in A} R_\alpha$  is represented by the completion  $R$  of  $R'$  in that topology, equipped with the natural morphisms  $i_\alpha : R_\alpha \rightarrow R$  deduced from the  $j_\alpha$ 's. Let  $r = (r_\alpha)_{\alpha \in A}, s = (s_\alpha)_{\alpha \in A} \in R$ . Then, for any open  $k$ -submodule  $U$  of  $R$  as before there is an index  $\alpha_0 \in A$  such that for any  $\alpha \geq \alpha_0$ ,  $r_\alpha - r_{\alpha_0}, s_\alpha - s_{\alpha_0} \in J_\alpha$ . So,

$$r_\alpha s_\alpha - r_{\alpha_0} s_{\alpha_0} = r_\alpha (s_\alpha - s_{\alpha_0}) + (r_\alpha - r_{\alpha_0}) s_{\alpha_0} \in J_\alpha.$$

This shows that  $(r_\alpha s_\alpha)_\alpha \in R$  so that we get a product map

$$R = \varinjlim_{\alpha \in A} R_\alpha \quad \times \quad R = \varinjlim_{\alpha \in A} R_\alpha \quad \longrightarrow \quad R = \varinjlim_{\alpha \in A} R_\alpha$$

$$(r = (r_\alpha)_{\alpha \in A} \quad , \quad s = (s_\alpha)_\alpha) \quad \longmapsto \quad rs = (r_\alpha s_\alpha)_\alpha$$

continuous for the product topology of  $R \times R$ . It is clear that  $R$  is in fact the colimit of the inductive system  $(R_\alpha)_{\alpha \in A}$  in  $\mathcal{LR}_k$ .

Assume now  $(R_\alpha)_{\alpha \in A}$  is an inductive system in  $\mathcal{LR}_k^u$ . The previous construction gives the inductive limit of  $(R_\alpha)_{\alpha \in A}$  in  $\mathcal{LR}_k$ . To construct explicitly the inductive limit of  $(R_\alpha)_{\alpha \in A}$  in  $\mathcal{LR}_k^u$  we repeat the construction of  $R'$  but endow it with the finest  $k$ -linear topology *weaker than the canonical topology* such that all  $j_\alpha$  are continuous. Then  $R^u$  is the completion of  $R'$  in that topology, and the existence of a product map

$$\mu_{R^u} : R^u \times R^u \longrightarrow R^u$$

follows.

Finally, let  $(R_\alpha)_{\alpha \in A}$  is an inductive system in  $\mathcal{RR}_k$ . To explicitly construct the inductive limit of  $(R_\alpha)_{\alpha \in A}$  in  $\mathcal{RR}_k$  we repeat the construction of  $R'$  but endow it with the finest *linear* topology such that all  $j_\alpha$  are continuous. A basis of open ideals of  $R'$  then consists of the ideals  $U$  such that  $j_\alpha^{-1}(U) = J_\alpha$  is an open ideal of  $R_\alpha$ , for any  $\alpha \in A$ . We prove as before that the completion  $R$  of  $R'$  in the latter topology is an object of  $\mathcal{RR}_k$  and that it represents the inductive limit of  $(R_\alpha)_{\alpha \in A}$  in  $\mathcal{RR}_k$ .

**Notation 5.4.** If  $\{R_\alpha\}_\alpha$  is an inductive system in  $\mathcal{RR}_k$  (resp.  $\mathcal{LR}_k^u$ , resp.  $\mathcal{LR}_k$ ), we denote by  $\varinjlim_\alpha^{\mathcal{R}} R_\alpha$  (resp.  $\varinjlim_\alpha^u R_\alpha$ , resp.  $\varinjlim_\alpha^{\mathcal{L}} R_\alpha$ ) its inductive limit in  $\mathcal{RR}_k$  (resp.  $\mathcal{LR}_k^u$ , resp.  $\mathcal{LR}_k$ ).

We introduce in the case of rings a notation analog to (2.13).

**Notation 5.5.** Let  $(R_\alpha)_{\alpha \in A}$  be an inductive system in  $\mathcal{LR}_k$  with transition morphisms  $j_{\alpha,\beta} : R_\alpha \rightarrow R_\beta$  for  $\alpha \leq \beta$ . For any  $\alpha \in A$  let  $\mathcal{P}(R_\alpha)$  denote the set of open ideals of  $R_\alpha$ . Then a *coherent system of open ideals* of  $(R_\alpha)_{\alpha \in A}$  is a system  $\mathcal{J} := (J_\alpha)_{\alpha \in A}$  such that for any  $\alpha \leq \beta$  in  $A$ ,  $j_{\alpha,\beta}^{-1}(J_\beta) = J_\alpha$ . The set  $\mathcal{C}((R_\alpha)_{\alpha \in A})$  of coherent systems of open ideals of  $(R_\alpha)_{\alpha \in A}$  forms a filter of  $k$ -submodules of  $\prod_{\alpha \in A} R_\alpha^{\text{for}}$ .

**Lemma 5.6.** *We use the notation of (5.5).*

1. *Let  $(R_\alpha)_{\alpha \in A}$  be an inductive system in  $\mathcal{LR}_k$ . Then*

$$(5.6.1) \quad \varinjlim_\alpha^{\mathcal{L}} R_\alpha = \varinjlim_{\mathcal{J} \in \mathcal{C}((R_\alpha)_{\alpha \in A})} \varinjlim_\alpha R_\alpha / J_\alpha$$

*where the inductive limit*

$$\varinjlim_\alpha R_\alpha / J_\alpha$$

*is taken in the category  $\mathcal{Mod}_k$ .*

2. *Let  $(R_\alpha)_{\alpha \in A}$  is an inductive system in  $\mathcal{LR}_k^u$ . Then*

$$(5.6.2) \quad \varinjlim_\alpha^u R_\alpha = \varinjlim_{I \in \mathcal{P}(k)} \varinjlim_{\alpha \in A}^u R_\alpha / \overline{IR_\alpha}$$

*where the inductive limit*

$$\varinjlim_{\alpha \in A}^u R_\alpha / \overline{IR_\alpha}$$

*is taken in the category  $\mathcal{LR}_{k/I}^u$ . It coincides as a topological  $k/I$ -module with the colimit of the same inductive system taken in the category  $\mathcal{LM}_{k/I}^u$  and also in the category  $\mathcal{LM}_{k/I} = \mathcal{LM}_{k/I}^u$  since  $k/I$  is discrete.*

3. *Let  $(R_\alpha)_{\alpha \in A}$  is an inductive system in  $\mathcal{RR}_k$ . Then*

$$(5.6.3) \quad \varinjlim_\alpha^{\mathcal{R}} R_\alpha = \varinjlim_{I \in \mathcal{P}(k)} \varinjlim_{\alpha \in A}^{\mathcal{R}} R_\alpha / \overline{IR_\alpha},$$

*where the inductive limit*

$$\varinjlim_{\alpha \in A}^{\mathcal{R}} R_\alpha / \overline{IR_\alpha}$$

*is taken in the category  $\mathcal{RR}_{k/I}$ .*

*Proof.* Clear. □

By general nonsense, for any inductive system  $\{R_\alpha\}_\alpha$  in  $\mathcal{RR}_k$ , there is a canonical morphism in  $\mathcal{LR}_k$

$$(5.6.4) \quad T : \varinjlim_\alpha^{\mathcal{L}} R_\alpha \longrightarrow \varinjlim_\alpha^{\mathcal{R}} R_\alpha.$$

For any object  $R$  in  $\mathcal{RR}_k$  the map

$$(5.6.5) \quad \begin{aligned} \mathrm{Hom}_{\mathcal{RR}_k}(\varinjlim_{\alpha}^{\mathcal{R}} R_{\alpha}, R) &\longrightarrow \mathrm{Hom}_{\mathcal{LR}_k}(\varinjlim_{\alpha}^{\mathcal{L}} R_{\alpha}, R) \\ \varphi &\longmapsto \varphi \circ T \end{aligned}$$

is in fact an isomorphism, since both source and target equal  $\varprojlim_{\alpha} \mathrm{Hom}_{\mathcal{RR}_k}(R_{\alpha}, R)$ .

**Lemma 5.7.**

1. *There exists a  $k$ -linear functor*

$$\mathcal{T} : \mathcal{IRR}_k \longrightarrow \mathcal{RR}_k \quad \text{such that} \quad \mathcal{T}(\varinjlim_{\alpha}^{\mathcal{L}} R_{\alpha}) = \varinjlim_{\alpha}^{\mathcal{R}} R_{\alpha},$$

*for any inductive system  $\{R_{\alpha}\}_{\alpha}$  in  $\mathcal{RR}_k$ .*

2. *Let*

$$(5.7.1) \quad \iota_{\mathcal{IRR}_k} : \mathcal{IRR}_k \hookrightarrow \mathcal{LR}_k \quad \text{and} \quad \iota_{\mathcal{RR}_k} : \mathcal{RR}_k \hookrightarrow \mathcal{LR}_k$$

*be the natural inclusions of full subcategories. There exists a natural transformation of functors  $\mathcal{IRR}_k \rightarrow \mathcal{LR}_k$*

$$(5.7.2) \quad \mathcal{S} : \iota_{\mathcal{IRR}_k} \longrightarrow \iota_{\mathcal{RR}_k} \circ \mathcal{T}$$

*such that, for any object  $\varinjlim_{\alpha}^{\mathcal{L}} R_{\alpha}$  of  $\mathcal{IRR}_k$*

$$(5.7.3) \quad \mathcal{S}(\varinjlim_{\alpha}^{\mathcal{L}} R_{\alpha}) : \varinjlim_{\alpha}^{\mathcal{L}} R_{\alpha} \longrightarrow \varinjlim_{\alpha}^{\mathcal{R}} R_{\alpha}$$

*coincides with the morphism  $T$  of (5.6.4), or, equivalently, is the image of  $\mathrm{id}_R$ , for  $R := \varinjlim_{\alpha} R_{\alpha}$ , via the identification of (5.6.5).*

*Proof.* The fact that the correspondence of objects

$$\mathcal{T} : \varinjlim_{\alpha}^{\mathcal{L}} R_{\alpha} \longmapsto \varinjlim_{\alpha}^{\mathcal{R}} R_{\alpha}$$

is well-defined follows from (5.6.5). The fact that  $\mathcal{T}$  extends to a functor, is general nonsense, and completes the proof of 1. Part 2 is self-explanatory.  $\square$

**Example 5.8.**

1. A typical example of an object of  $\mathcal{LR}_{\mathbb{Z}_p}$ , but not of  $\mathcal{RR}_{\mathbb{Z}_p}$ , is any non-archimedean non-trivially valued field extension  $K$  of  $(\mathbb{Q}_p, v_p)$ . This  $K$  is also an object of  $\mathcal{LR}_{K^{\circ}}$  but not of  $\mathcal{RR}_{K^{\circ}}$ . The same situation occurs for any commutative  $K$ -Banach algebra.
2. Let  $k = \mathbb{Z}_p$  and let  $\mathbb{Z}_p\{x\}$  be the  $p$ -adic completion of  $\mathbb{Z}_p[x]$ . So,  $\mathbb{Z}_p\{x\}$  is an object of  $\mathcal{RR}_k$ . Let  $F : \mathbb{Z}_p\{x\} \rightarrow \mathbb{Z}_p\{x\}$  be the  $\mathcal{RR}_k$ -morphism such that  $F(x) = px$ . Consider the inductive system

$$(\mathbb{Z}_p\{x\}, F) := \mathbb{Z}_p\{x\} \xrightarrow{F} \mathbb{Z}_p\{x\} \xrightarrow{F} \dots$$

in  $\mathcal{RR}_k$ . Then

$$\varinjlim^{\mathcal{R}} (\mathbb{Z}_p\{x\}, F) = \varinjlim^u (\mathbb{Z}_p\{x\}, F) = \mathbb{Z}_p \quad \text{while} \quad \varinjlim^{\mathcal{L}} (\mathbb{Z}_p\{x\}, F) = \mathbb{Q}_p\{x\}.$$

## 6 Tensor product of rings

For two objects  $A, B$  of  $\mathcal{LR}_k^u$ , the  $k$ -module  $A \widehat{\otimes}_k^u B$  is naturally an object of  $\mathcal{LR}_k^u$  with product

$$(6.0.1) \quad \begin{aligned} A \widehat{\otimes}_k^u B \times A \widehat{\otimes}_k^u B &\longrightarrow A \widehat{\otimes}_k^u B \\ (a_1 \widehat{\otimes}_k^u b_1, a_2 \widehat{\otimes}_k^u b_2) &\longmapsto a_1 a_2 \widehat{\otimes}_k^u b_1 b_2 . \end{aligned}$$

If, in particular,  $A, B$  are objects of  $\mathcal{RR}_k$ , so is  $A \widehat{\otimes}_k^u B$ . Moreover, for any object  $R$  of  $\mathcal{LR}_k^u$  the product  $R \times R \rightarrow R$  factors through a morphism

$$\mu_R : R \widehat{\otimes}_k^u R \rightarrow R$$

in  $\mathcal{LM}_k^u$ . This holds in particular if  $R$  is in  $\mathcal{RR}_k$ . Similarly, for two objects  $A, B$  of  $\mathcal{LR}_k$ , the  $k$ -module  $A \widehat{\otimes}_k^c B$  is naturally an object of  $\mathcal{LR}_k$  with a similar formula for the product. For any object  $R$  of  $\mathcal{LR}_k$  the product  $R \times R \rightarrow R$  factors through a morphism

$$\mu_R : R \widehat{\otimes}_k^c R \rightarrow R$$

in  $\mathcal{LM}_k$ .

**Definition 6.1.** For objects  $A, B, C$  of  $\mathcal{LR}_k$  we denote by  $\text{Ril}_k^c(A \times B, C)$  (resp.  $\text{Ril}_k^u(A \times B, C)$ ) the  $k$ -submodule of  $\text{Bil}_k^c(A \times B, C)$  (resp.  $\text{Bil}_k^u(A \times B, C)$ ) consisting of functions  $(a, b) \mapsto \varphi(a, b)$  such that for any  $a \in A$  (resp.  $b \in B$ ) the map  $b \mapsto \varphi(1, b)$  (resp.  $a \mapsto \varphi(a, 1)$ ) is a morphism in  $\mathcal{LR}_k$  and

$$\varphi(a, b) = \varphi(a, 1)\varphi(1, b) .$$

For any objects  $A, B$  of  $\mathcal{LR}_k$ , we consider the functors

$$(6.1.1) \quad \text{Ril}_k^c(A \times B, -) : \mathcal{LR}_k \longrightarrow \text{Mod}_k ,$$

$$(6.1.2) \quad \text{Ril}_k^u(A \times B, -) : \mathcal{LR}_k \longrightarrow \text{Mod}_k .$$

**Lemma 6.2.**

1. For any  $A, B$  in  $\mathcal{LR}_k$ , the functor  $\text{Ril}_k^c(A \times B, -)$  of (6.1.1) is represented by the object  $A \widehat{\otimes}_k^c B$  of  $\mathcal{LR}_k$ .
2. For any  $A, B$  in  $\mathcal{LR}_k$ , the functor  $\text{Ril}_k^u(A \times B, -)$  of (6.1.2) is represented by the object  $A \widehat{\otimes}_k^u B$  of  $\mathcal{LR}_k$ .
3. If  $A$  and  $B$  are objects of  $\mathcal{LR}_k^u$  (resp.  $\mathcal{RR}_k$ ), then  $A \widehat{\otimes}_k^u B$  is an object of  $\mathcal{LR}_k^u$  (resp. of  $\mathcal{RR}_k$ ).
4. For any  $A, B$  in  $\mathcal{LR}_k$ , there is a canonical morphism

$$(6.2.1) \quad \Phi : A \widehat{\otimes}_k^c B \longrightarrow A \widehat{\otimes}_k^u B$$

such that, for any  $R$  in  $\mathcal{LR}_k^u$ ,  $\varphi \mapsto \varphi \circ \Phi$  is the natural inclusion

$$\text{Ril}_k^u(A \times B, R) \longrightarrow \text{Ril}_k^c(A \times B, R) .$$

*Proof.* The first and the second assertions of the lemma are proved similarly; we only prove the second.

We recall that (for any  $A$  and  $B$  in  $\mathcal{LM}_k$ )  $A \widehat{\otimes}_k^u B$  is the separated completion of  $A \otimes_k B$  (meaning of  $A^{\text{for}} \otimes_{k^{\text{for}}} B^{\text{for}}$ ) equipped with a suitable  $k$ -linear topology. For any  $X$  in  $\mathcal{LM}_k$  we have an isomorphism of  $k^{\text{for}}$ -modules

$$(6.2.2) \quad \begin{aligned} \text{Bil}_k^u(A \times B, X) &\xrightarrow{\sim} \text{Hom}_{\mathcal{LM}_k}(A \widehat{\otimes}_k^u B, X) \\ \varphi &\longmapsto (\Phi : a \otimes_k b \mapsto \varphi(a, b)) . \end{aligned}$$

In the present situation, we also have a ring structure on  $A \widehat{\otimes}_k^u B$  and maps  $j_1 : A \rightarrow A \widehat{\otimes}_k^u B$  (resp.  $j_2 : B \rightarrow A \widehat{\otimes}_k^u B$ ) which extend by continuity  $a \mapsto a \otimes_k 1$  (resp.  $b \mapsto 1 \otimes_k b$ ) and are morphisms in  $\mathcal{LR}_k$ . Then the identification (6.2.2) is characterized by the condition that  $\Phi \circ j_1 : A \rightarrow X$  is  $a \mapsto \varphi(a, 1)$  while  $\Phi \circ j_2 : B \rightarrow X$  is  $b \mapsto \varphi(1, b)$ . If  $X$  is an object of  $\mathcal{LR}_k$  the identification (6.2.2) restricts to an identification

$$\text{Ril}_k^u(A \times B, X) = \text{Hom}_{\mathcal{LR}_k}(A \widehat{\otimes}_k^u B, X) .$$

The other assertions of the statement are clear. □

**Proposition 6.3.**

1. Let  $\{A_\alpha\}_\alpha$  and  $\{B_\beta\}_\beta$  be inductive systems in  $\mathcal{LR}_k$ . Then

$$(6.3.1) \quad \varinjlim_\alpha {}^{\mathcal{L}} A_\alpha \widehat{\otimes}_k^c \varinjlim_\beta {}^{\mathcal{L}} B_\beta = \varinjlim_{\alpha, \beta} {}^{\mathcal{L}} (A_\alpha \widehat{\otimes}_k^c B_\beta) .$$

2. Let  $\{A_\alpha\}_\alpha$  and  $\{B_\beta\}_\beta$  be projective systems in  $\mathcal{LR}_k$ . Then

$$(6.3.2) \quad \varprojlim_\alpha A_\alpha \widehat{\otimes}_k^u \varprojlim_\beta B_\beta = \varprojlim_{\alpha, \beta} A_\alpha \widehat{\otimes}_k^u B_\beta .$$

*Proof.* The two statements follow from (3.5.1) and (3.5.2), respectively. □

## 7 Base change

**Lemma 7.1.** Let  $A$  be any object of  $\mathcal{RR}_k$ .

1. Let  $M$  be an object of  $\mathcal{LM}_k$  (resp. of  $\mathcal{LM}_k^u$ ). The map

$$A \times (A \otimes_k M) \rightarrow A \otimes_k M , \quad (a, b \otimes m) \mapsto ab \otimes m ,$$

extends to an  $A$ -bilinear map

$$A \times (A \widehat{\otimes}_k^c M) \rightarrow A \widehat{\otimes}_k^c M$$

(resp.

$$A \times (A \widehat{\otimes}_k^u M) \rightarrow A \widehat{\otimes}_k^u M )$$

(resp. uniformly) continuous for the product topology (resp. uniformity) of  $A \times (A \widehat{\otimes}_k^c M)$  (resp.  $A \times (A \widehat{\otimes}_k^u M)$ ) which makes  $A \widehat{\otimes}_k^c M$  (resp.  $A \widehat{\otimes}_k^u M$ ) into an  $A$ -linearly topologized (resp. uniform) separated and complete  $A$ -module. The correspondence

$$(7.1.1) \quad \begin{aligned} (-)_A^c : \mathcal{LM}_k &\longrightarrow \mathcal{LM}_A \\ M &\longmapsto (M)_A^c := A \widehat{\otimes}_k^c M \end{aligned}$$

(resp.

$$(7.1.2) \quad \begin{aligned} (-)_A^u : \mathcal{LM}_k^u &\longrightarrow \mathcal{LM}_A^u \\ M &\longmapsto (M)_A^u := A \widehat{\otimes}_k^u M \end{aligned}$$

is part of an additive functor which we call continuous (resp. uniform) extension of scalars by  $A$ . The functor  $(-)_A^c$  (resp.  $(-)_A^u$ ) commutes with inductive (resp. projective) limits in  $\mathcal{LM}_k$  and  $\mathcal{LM}_A$  (resp. in  $\mathcal{LM}_k^u$  and  $\mathcal{LM}_A^u$ ). The functor  $(-)_A^c$  (resp.  $(-)_A^u$ ) is left-adjoint to the natural inclusion functor  $\mathcal{LM}_A \hookrightarrow \mathcal{LM}_k$  (resp.  $\mathcal{LM}_A^u \hookrightarrow \mathcal{LM}_k^u$ ). Namely, for any  $M$  in  $\mathcal{LM}_k$  and  $N$  in  $\mathcal{LM}_A$  (resp. for any  $M$  in  $\mathcal{LM}_k^u$  and  $N$  in  $\mathcal{LM}_A^u$ )

$$(7.1.3) \quad \text{Hom}_{\mathcal{LM}_k}(M, N) = \text{Hom}_{\mathcal{LM}_A}((M)_A^c, N)$$

(resp.

$$(7.1.4) \quad \text{Hom}_{\mathcal{LM}_k^u}(M, N) = \text{Hom}_{\mathcal{LM}_A^u}((M)_A^u, N) .$$

2. For any  $B$  in  $\mathcal{LR}_k$  (resp. in  $\mathcal{LR}_k^u$ , resp. in  $\mathcal{RR}_k$ ), the canonical morphism  $A \rightarrow A \widehat{\otimes}_k^c B$ ,  $a \mapsto a \otimes 1$ , makes  $A \widehat{\otimes}_k^c B$  into an  $A$ -linearly (resp. a uniform  $A$ -linearly, resp. a linearly) topologized separated and complete  $A$ -ring. Moreover

$$(7.1.5) \quad \begin{aligned} (-)_A^c : \mathcal{LR}_k &\longrightarrow \mathcal{LR}_A \\ B &\longmapsto (B)_A^c := A \widehat{\otimes}_k^c B \end{aligned}$$

(resp.

$$(7.1.6) \quad \begin{aligned} (-)_A^u : \mathcal{LR}_k^u &\longrightarrow \mathcal{LR}_A^u \\ B &\longmapsto (B)_A^u := A \widehat{\otimes}_k^u B , \end{aligned}$$

resp.

$$(7.1.7) \quad \begin{aligned} (-)_A^u : \mathcal{RR}_k &\longrightarrow \mathcal{RR}_A \\ B &\longmapsto (B)_A^u := A \widehat{\otimes}_k^u B \end{aligned}$$

is an additive functor, commuting with inductive limits in  $\mathcal{LR}_k$  and  $\mathcal{LR}_A$  (resp. with projective limits in  $\mathcal{LR}_k^u$  and  $\mathcal{LR}_A^u$ , resp. with projective limits in  $\mathcal{RR}_k$  and  $\mathcal{RR}_A$ ), which we call continuous (resp. uniform, resp. uniform) base-change by  $A$ . The functor  $(-)_A^c$  (resp.  $(-)_A^u$ , resp.  $(-)_A^u$ ) is left-adjoint to the natural inclusion functor  $\mathcal{LR}_A \hookrightarrow \mathcal{LR}_k$  (resp.  $\mathcal{LR}_A^u \hookrightarrow \mathcal{LR}_k^u$ , resp.  $\mathcal{RR}_A \hookrightarrow \mathcal{RR}_k$ ). Namely, for any  $R$  in  $\mathcal{LR}_k$  and  $S$  in  $\mathcal{LR}_A$  (resp. for any  $R$  in  $\mathcal{LR}_k^u$  and  $S$  in  $\mathcal{LR}_A^u$ , resp. for any  $R$  in  $\mathcal{RR}_k$  and  $S$  in  $\mathcal{RR}_A$ )

$$(7.1.8) \quad \text{Hom}_{\mathcal{LR}_k}(R, S) = \text{Hom}_{\mathcal{LR}_A}((R)_A^c, S)$$

(resp.

$$(7.1.9) \quad \text{Hom}_{\mathcal{LR}_k^u}(R, S) = \text{Hom}_{\mathcal{LR}_A^u}((R)_A^u, S) ,$$

resp.

$$(7.1.10) \quad \text{Hom}_{\mathcal{RR}_k}(R, S) = \text{Hom}_{\mathcal{RR}_A}((R)_A^u, S) .$$

In particular,  $(-)_A^u : \mathcal{RR}_k \rightarrow \mathcal{RR}_A$  commutes with direct limits.

*Proof.* We prove the adjunction property for  $(-)^u_A$ . Let  $M$  (resp.  $N$ ) be an object of  $\mathcal{LM}_k^u$  (resp.  $\mathcal{LM}_A^u$ ). Then from (2.25.4) we get

$$(7.1.11) \quad \text{Hom}_{\mathcal{LM}_A^u}(A \widehat{\otimes}_k^u M, N) = \varprojlim_{Q \in \mathcal{P}(N)} \varinjlim_{P \in \mathcal{P}(M)} \text{Hom}_A(M/P \otimes_k A, N/Q)$$

where  $\text{Hom}_A(M/P \otimes_k A, N/Q) = \text{Hom}_{A/J}(M/P \otimes_k A/J, N/Q)$  for any  $J \in \mathcal{P}(A)$  such that  $JN \subset Q$  and  $(J \cap k)M \subset P$ . If  $J \cap k \supset I \in \mathcal{P}(k)$  so that  $IM \subset P$ , the latter equals  $\text{Hom}_{k/I}(M/P, N/Q)$  and we conclude by (2.25.4).

We now prove the adjunction property for  $(-)^c_A$ . Let  $M$  (resp.  $N$ ) be an object of  $\mathcal{LM}_k$  (resp.  $\mathcal{LM}_A$ ). Then

$$(7.1.12) \quad \begin{aligned} \text{Hom}_{\mathcal{LM}_A}(A \widehat{\otimes}_k^c M, N) &= \{ \varphi \in \text{Hom}_{\mathcal{LM}_k}(A \widehat{\otimes}_k^c M, N) \mid \varphi \text{ is } A\text{-linear} \} = \\ &= \{ \varphi \in \text{Bil}_k^c(A \times M, N) \mid \varphi \text{ is } A\text{-linear in the first variable} \} = \text{Hom}_{\mathcal{LM}_k}(M, N). \end{aligned}$$

□

**Definition 7.2.** Let  $f : A \rightarrow B$  be a morphism in  $\mathcal{RR}_k$ . We say that  $f$  is *pro-flat* or that  $B$  is *pro-flat over  $A$*  if, for any open ideal  $J$  in  $B$ ,  $B/J$  is a flat  $A/f^{-1}(J)$ -module.

**Lemma 7.3.** Let  $M$  and  $N$  be objects of  $\mathcal{LM}_k$  endowed with the  $k$ -canonical topology (so, in particular,  $M$  and  $N$  are uniform) and let  $A$  be an object of  $\mathcal{RR}_k$ .

1.  $M \widehat{\otimes}_k^u N$  carries the  $k$ -canonical topology. If  $k$  satisfies **OPW** then  $M \widehat{\otimes}_k^c N = M \widehat{\otimes}_k^u N$ .
2. The morphism  $(M)_A^c \rightarrow (M)_A^u$  is an isomorphism of  $\mathcal{LM}_A^u$  and both objects carry the  $A$ -canonical topology.
3. Let  $I$  (resp.  $J$ ) be an open ideal of  $k$  (resp.  $A$ ) such that  $IA \subset J$ . If  $M/\overline{IM}$  is a flat  $k/I$ -module,  $(M)_A^u/\overline{J(M)_A^u} = (M)_A^c/\overline{J(M)_A^c}$  is a flat  $A/J$ -module.
4. Assume  $A$  is pro-flat over  $k$  and let  $L$  be an open sub-object of  $M$ . Then  $(L)_A^u$  is an open sub-object of  $(M)_A^u$ .
5. If  $A$  is pro-flat over  $k$ , for any open sub-object  $L$  of  $M$ ,  $(L)_A^u = (L)_A^c$  is an open sub-object of  $(M)_A^u = (M)_A^c$  and all these objects carry the  $A$ -canonical topology.

*Proof.*

1. According to (3.3.2) it suffices to show that

$$C := \text{Ker}(M \widehat{\otimes}_k^u N \rightarrow M/\overline{IM} \otimes_k N/\overline{IN})$$

is the closure of  $I(M \widehat{\otimes}_k^u N)$  in  $M \widehat{\otimes}_k^u N$ . By [4, II, §3, n. 6, Cor. 1 of Prop. 6], the kernel of the morphism

$$M^{\text{for}} \otimes_{k^{\text{for}}} N^{\text{for}} \longrightarrow M^{\text{for}}/IM^{\text{for}} \otimes_{k^{\text{for}}} N^{\text{for}}/IN^{\text{for}}$$

is

$$C' := M^{\text{for}} \otimes_{k^{\text{for}}} IN^{\text{for}} + IM^{\text{for}} \otimes_{k^{\text{for}}} N^{\text{for}} = I(M^{\text{for}} \otimes_{k^{\text{for}}} N^{\text{for}}).$$

So,  $C$  coincides with the closure of  $C'$  in  $M \widehat{\otimes}_k^u N$  and therefore also with the closure of  $I(M \widehat{\otimes}_k^u N)$  in  $M \widehat{\otimes}_k^u N$ , as claimed.

If  $k$  satisfies **OPW**, then the ideals of the form  $IJ$ , for  $I, J \in \mathcal{P}(k)$  are open. So the system of  $k$ -submodules  $IM \otimes_k JN$ , for  $I, J \in \mathcal{P}(k)$ , is cofinal with the system  $IM \otimes_k N$ , for  $I \in \mathcal{P}(k)$ . So,  $M \widehat{\otimes}_k^c N$  coincides with  $M \widehat{\otimes}_k^u N$ .

2.  $(M)_A^c$  is the completion of  $A \otimes_k M$  in the  $k$ -linear topology with fundamental system of open  $k$ -submodules given by the family of the  $\text{Im}(J \otimes IM)$ , where  $I$  (resp.  $J$ ) runs over a fundamental system of open ideals of  $k$  (resp.  $A$ ). This equals the fundamental system of



open  $A$ -submodules  $\{\text{Im}((J + I A) \otimes M)\}_{I,J}$  for  $I, J$  as before. A cofinal system  $\mathcal{F}$  is given by the condition  $I A \subset J$ , that is by

$$\mathcal{F} = \{\text{Im}((J + I A) \otimes M)\}_{I \subset J} = \{\text{Im}(J \otimes_k M)\}_J = \{J(A \otimes_k M)\}_J,$$

so that the topology obtained on  $(M)_A^c$  is indeed the  $A$ -canonical one. Similarly for  $(M)_A^u$ .

3. Let  $J$  be any open ideal of  $A$ , and let  $I$  be an open ideal of  $k$  such that  $I A \subset J$ . Then

$$(M)_A^c / \overline{J(M)_A^c} = (M)_A^c / \overline{J \otimes_k M} = (M)_A^c / \overline{J \otimes_k M + A \otimes_k I M} = A/J \otimes_{k/I} M / \overline{I M}$$

is a flat  $A/J$ -module. Similarly for  $(M)_A^u / \overline{J(M)_A^u}$ .

4. The open sub-object  $L$  of  $M$  may be described as a projective system of submodules  $\overline{L}_I \subset M / \overline{I M}$ , for  $I \in \mathcal{P}(k)$ , such that the maps  $\overline{L}_J \rightarrow \overline{L}_I$ , for  $J \subset I$  in  $\mathcal{P}(k)$ , are surjective for sufficiently small  $I$ . This properties are inherited by the projective system  $\overline{L}_I \otimes_{k/I} A/H$  for any  $H \in \mathcal{P}(A)$ , where  $\overline{L}_I \otimes_{k/I} A/H \rightarrow M / \overline{I M} \otimes_{k/I} A/H$  is injective since  $A$  is pro-flat. The projective system  $\overline{L}_I \otimes_{k/I} A/H$  then defines an open sub-object of  $(M)_A^u$ . Since  $(-)_A^u$  commutes with projective limits, the open sub-object of  $(M)_A^u$  which we get is  $(L)_A^u$ .

5. This is simply a summary of what we have proven, taking into account Remark 2.28.  $\square$

**Proposition 7.4.** *We assume that  $k$  satisfies condition **OPW**. Let  $M$  and  $N$  be objects of  $\mathcal{PB}_k$ , with gauges  $\mathcal{G}(M)$  and  $\mathcal{G}(N)$ , respectively. Then  $M \widehat{\otimes}_k^c N$  is an object of  $\mathcal{PB}_k$  with gauge*

$$\{P \widehat{\otimes}_k^c Q = P \widehat{\otimes}_k^u Q\}_{P,Q}$$

where  $P \in \mathcal{G}(M)$ ,  $Q \in \mathcal{G}(N)$ . In particular,

$$(7.4.1) \quad M \widehat{\otimes}_k^c N = \varinjlim_{P,Q} P \widehat{\otimes}_k^u Q.$$

*Proof.* The case of  $k$  discrete follows from Remark 4.5. We then assume that  $k$  is not discrete. The fact that  $P \widehat{\otimes}_k^c Q = P \widehat{\otimes}_k^u Q$  is 1 of Lemma 7.3. We prove the first part of the statement. We first need to show that

**Lemma 7.5.** *Any morphism  $P \widehat{\otimes}_k^u Q \rightarrow P' \widehat{\otimes}_k^u Q'$  in (7.4.1), for  $P \subset P'$  in  $\mathcal{G}(M)$  and  $Q \subset Q'$  in  $\mathcal{G}(N)$ , is an open embedding. So,  $P \widehat{\otimes}_k^u Q$  is an open sub-object of  $P' \widehat{\otimes}_k^u Q'$  carrying the canonical topology.*

*Proof.* From 1 of Lemma 7.3, we know that both source and target carry the canonical topology. To show that  $P \widehat{\otimes}_k^u Q \rightarrow P' \widehat{\otimes}_k^u Q'$  is an open sub-object we first need to show that, for any open ideal  $I$  of  $k$ , the map

$$P \widehat{\otimes}_k^u Q / \overline{I(P \widehat{\otimes}_k^u Q)} \rightarrow P' \widehat{\otimes}_k^u Q' / \overline{I(P' \widehat{\otimes}_k^u Q')}$$

is injective. But this map coincides with

$$(7.5.1) \quad P / \overline{I P} \otimes_{k/I} Q / \overline{I Q} \rightarrow P' / \overline{I P'} \otimes_{k/I} Q' / \overline{I Q'}.$$

The latter is injective for the following reasons. First of all both maps

$$P / \overline{I P} \rightarrow P' / \overline{I P'} \quad \text{and} \quad Q / \overline{I Q} \rightarrow Q' / \overline{I Q'}$$

are injective, because the subspace topology of  $P$  in  $P'$  (resp. of  $Q$  in  $Q'$ ) is the canonical topology of  $P$  (resp.  $Q$ ), which means that  $\overline{I P'} \cap P = \overline{I P}$  (resp.  $\overline{I Q'} \cap Q = \overline{I Q}$ ). Then, by

the flatness assumptions, the map (7.5.1) is injective. We then need to check that for any  $J \subset I$  in  $\mathcal{P}(k)$ , and for  $I$  sufficiently small,

$$P/\overline{JP} \otimes_{k/J} Q/\overline{JQ} \rightarrow P/\overline{IP} \otimes_{k/I} Q/\overline{IQ}$$

and

$$P'/\overline{J'P} \otimes_{k/J} Q'/\overline{J'Q'} \rightarrow P'/\overline{I'P'} \otimes_{k/I} Q'/\overline{I'Q'}$$

are both surjective. But this is clear by right-exactness of the tensor product in  $\mathcal{M}od_k$ . This implies at the same time that the open sub-object  $P \widehat{\otimes}_k^u Q$  of  $P' \widehat{\otimes}_k^u Q'$  carries the canonical topology.  $\square$

We now get back to the proof of Proposition 7.4. Condition 1 of Definition 4.1 holds by Lemma 7.5. Condition 2 holds because, for any open ideal  $I$  of  $k$ ,

$$P \widehat{\otimes}_k^u Q / \overline{IP \widehat{\otimes}_k^u Q + P \widehat{\otimes}_k^u IQ} = P/\overline{IP} \otimes_{k/I} Q/\overline{IQ}$$

is a flat  $k/I$ -module. Part 3 of that definition follows from (7.4.1). But (7.4.1) follows from (3.5.1) and the fact that  $P \widehat{\otimes}_k^u Q = P \widehat{\otimes}_k^c Q$  proven in 1 of Lemma 7.3.  $\square$

**Proposition 7.6.** *Let  $A$  be an object of  $\mathcal{RR}_k$  and  $M$  be an object of  $\mathcal{PB}_k$ .*

1. *If  $A$  is pro-flat over  $k$ ,  $(M)_A^c$  is an object of  $\mathcal{PB}_A$ .*
2. *If  $k$  satisfies **OPW**, then  $(\mathcal{PB}_k, \widehat{\otimes}_k^c)$  is a monoidal category.*
3. *If both  $k$  and  $A$  satisfy **OPW** and  $A$  is pro-flat over  $k$ , the functor  $M \mapsto (M)_A^c$  induces an additive functor of monoidal categories*

$$(\mathcal{PB}_k, \widehat{\otimes}_k^c) \rightarrow (\mathcal{PB}_A, \widehat{\otimes}_A^c).$$

*Proof.* Let  $M$  be an object of  $\mathcal{PB}_k$  and  $\mathcal{G}$  be a  $k$ -gauge of  $M$ . We want to show that the family

$$(\mathcal{G})_A^c := \{(P)_A^c = (P)_A^u \mid P \in \mathcal{G}\}$$

is an  $A$ -gauge in  $(M)_A^c$ . For any  $P \in \mathcal{G}$ , let  $i_P : P \rightarrow M$  be the open embedding of  $P$  in  $M$ . The fact that  $(i_P)_A^c : (P)_A^c \rightarrow (M)_A^c$  is an open sub-object follows from the fact that if  $i_{P,P'} : P \rightarrow P'$  is the open embedding of  $P$  in  $P'$ , for  $P' \supset P$  in  $\mathcal{G}$ , then  $(i_P)_A^c : (P)_A^c \rightarrow (P')_A^c$  is an open embedding as shown in part 4 of Lemma 7.3. In fact, as indicated in Lemma 7.1 and in part 1 of Proposition 6.3, the functor  $(-)_A^c$  of (7.1.1) commutes to inductive limits in  $\mathcal{LM}_k$  and  $\mathcal{LM}_A$ . So,  $(M)_A^c = \varinjlim_{P \in \mathcal{G}} (P)_A^c$  is a filtering inductive limit of open embeddings,

hence it is the increasing union of the open sub-objects  $(P)_A^c$ , for  $P \in \mathcal{G}$ . Conditions 1 and 2 of Definition 4.1 follow from parts 2 and 3 of Lemma 7.3, respectively. Condition 3 has already been proven.

Part 2 follows from part 1 of Lemma 7.3, taking into account part 1 of Proposition 6.3.

Part 3 is clear.  $\square$

## 8 Representable functors

**Definition 8.1.** *We will denote by  $\mathcal{IRR}_k$  the full subcategory of  $\mathcal{LR}_k$  whose objects  $R = \varinjlim_{\alpha}^{\mathcal{L}} R_{\alpha} = \varinjlim_{\alpha}^u R_{\alpha}$  are inductive limits in both  $\mathcal{LR}_k^u$  and  $\mathcal{LR}_k$  of the same inductive system  $\{R_{\alpha}\}_{\alpha \in A}$  in  $\mathcal{RR}_k$ .*

**Remark 8.2.**  $\mathcal{IRR}_k$  is in fact a full subcategory of  $\mathcal{LR}_k^u$ , and its objects are those inductive limits in  $\mathcal{LR}_k$  of inductive systems  $\{R_{\alpha}\}_{\alpha \in A}$  in  $\mathcal{RR}_k$ , which happen to be objects of  $\mathcal{LR}_k^u$ .

**Lemma 8.3.** *Let  $\{R_\alpha\}_{\alpha \in A}$  be an inductive system in  $\mathcal{RR}_k$ . Assume*

$$\varinjlim_{\alpha}^{\mathcal{L}} R_\alpha = \varinjlim_{\alpha}^u R_\alpha .$$

*Then for any object  $C$  of  $\mathcal{RR}_k$*

$$\varinjlim_{\alpha}^{\mathcal{L}} (R_\alpha)_C^u = \varinjlim_{\alpha}^u (R_\alpha)_C^u .$$

*Proof.* We need to show that  $\varinjlim_{\alpha}^{\mathcal{L}} (R_\alpha)_C^u$  is  $C$ -uniform. By assumption, we have an isomorphism

$$\varprojlim_{\mathcal{J} \in \mathcal{C}(\{R_\alpha\}_{\alpha \in A})} \varinjlim_{\alpha} R_\alpha / J_\alpha \xrightarrow{\sim} \varprojlim_{I \in \mathcal{P}(k)} \varinjlim_{\alpha} R_\alpha / \overline{IR_\alpha}$$

in  $\mathcal{LR}_k^u$ . So, for any coherent system of open ideals  $\mathcal{J} = (J_\alpha)_\alpha$  there exists an open ideal  $I \in \mathcal{P}(k)$  such that  $J_\alpha \supset IR_\alpha$ , for any  $\alpha \in A$ .

Let now  $\mathcal{H} = (H_\alpha)_\alpha$  be a coherent system of open ideals in the inductive system  $(R_\alpha \widehat{\otimes}_k^u C)_{\alpha \in A}$  in  $\mathcal{RR}_C$ . We need to prove that the canonical map provides an isomorphism

$$(8.3.1) \quad \begin{aligned} & \varprojlim_{\mathcal{H} \in \mathcal{C}((R_\alpha \widehat{\otimes}_k^u C)_{\alpha \in A})} \varinjlim_{\alpha} (R_\alpha \widehat{\otimes}_k^u C) / H_\alpha \xrightarrow{\sim} \\ & \varprojlim_{J \in \mathcal{P}(C)} \varinjlim_{\alpha} (R_\alpha \widehat{\otimes}_k^u C / J) = \varprojlim_{J \in \mathcal{P}(C)} \varinjlim_{\alpha} ((R_\alpha / \overline{(J \cap k)R_\alpha}) \otimes_{k/(J \cap k)} C / J) . \end{aligned}$$

Again (8.3.1) will hold if for any  $\mathcal{H} = (H_\alpha)_\alpha$  as before, there exists an open ideal  $J \in \mathcal{P}(C)$  such that  $H_\alpha \supset R_\alpha \widehat{\otimes}_k^u J$ , for any  $\alpha \in A$ . Equivalently, we may replace  $H_\alpha$  by its inverse image in  $R_\alpha \otimes_k C \rightarrow R_\alpha \widehat{\otimes}_k^u C$ . We keep the name  $H_\alpha$  for that inverse image. Now, for any  $\alpha \in A$ ,

$$R_\alpha \widehat{\otimes}_k^u C = \varprojlim_{J \in \mathcal{P}(R_\alpha), Q \in \mathcal{P}(C)} R_\alpha / J \otimes_k C / Q$$

so that we may assume that  $H_\alpha$  has the form

$$H_\alpha = J_\alpha \otimes C + R_\alpha \otimes_k Q_\alpha$$

for some  $J_\alpha \in \mathcal{P}(R_\alpha)$  and  $Q_\alpha \in \mathcal{P}(C)$ . The coherence condition becomes

$$(j_{\alpha,\beta} \otimes_k C)^{-1} (J_\beta \otimes C + R_\beta \otimes_k Q_\beta) = J_\alpha \otimes C + R_\alpha \otimes_k Q_\alpha$$

for any  $\alpha \leq \beta$ . But  $j_{\alpha,\beta} \otimes_k C$  is the continuous  $C$ -linear extension of  $j_{\alpha,\beta} : R_\alpha \rightarrow R_\beta$ , so that

$$(j_{\alpha,\beta} \otimes_k C)^{-1} (J_\beta \otimes C + R_\beta \otimes_k Q_\beta) = j_{\alpha,\beta}^{-1} (J_\alpha) \widehat{\otimes}_k^u C + R_\alpha \widehat{\otimes}_k^u Q_\beta .$$

This shows that there is an ideal  $Q \in \mathcal{P}(C)$  such that  $H_\alpha \supset R_\alpha \otimes_k Q$ , for any  $\alpha$ . On the other hand,  $\mathcal{J} := (J_\alpha)_\alpha$  is a coherent system of open ideals in  $(R_\alpha)_\alpha$ , so that, as we saw before, there exists an open ideal  $I \in \mathcal{P}(k)$  such that  $IR_\alpha \subset J_\alpha$  for any  $\alpha$ . We conclude that for any coherent system of open ideals  $\mathcal{H} = (H_\alpha)_\alpha$  as before,

$$H_\alpha \supset IR_\alpha \otimes_k C + R_\alpha \otimes_k Q .$$

□

An object  $\mathcal{L}_k = \varinjlim_{\alpha \in A}^{\mathcal{L}} R_\alpha$  of  $\mathcal{IRR}_k$  determines the functor  $F : \mathcal{LR}_k \rightarrow \mathcal{Mod}_k$  given by

$$(8.3.2) \quad X \mapsto \text{Hom}_{\mathcal{LR}_k}(\varinjlim_{\alpha \in A} R_\alpha, X) = \varprojlim_{\alpha \in A} \text{Hom}_{\mathcal{LR}_k}(R_\alpha, X) .$$

We say that  $\mathcal{L}_k$  represents the functor  $F$ . Recall that  $\{(R_\alpha)_{\alpha \in A}, (\iota_{\alpha,\beta} : R_\alpha \rightarrow R_\beta)_{\alpha \leq \beta}\}$  is an inductive system in  $\mathcal{RR}_k$  and that  $\mathcal{L}_k$  is an object of  $\mathcal{LR}_k^u$ . By Yoneda's lemma,  $\mathcal{L}_k$  is determined by the restriction of the functor  $F$  to  $\mathcal{LR}_k^u$ .

**Proposition 8.4.** *Let  $F : \mathcal{LR}_k \rightarrow \text{Mod}_k$  be defined by (8.3.2) as above. Then, for any object  $C$  of  $\mathcal{RR}_k$  the restriction of  $F$  to  $\mathcal{LR}_C^u$  is represented by the object*

$$\mathcal{L}_C := \varinjlim_{\alpha \in A} {}^{\mathcal{L}}(C \widehat{\otimes}_k^u R_\alpha) = \varinjlim_{\alpha \in A} {}^u(C \widehat{\otimes}_k^u R_\alpha)$$

of  $\mathcal{IRR}_C$ .

*Proof.* In fact, any object  $X$  of  $\mathcal{LR}_C^u$  may be viewed as an object of  $\mathcal{LR}_k^u$ . We have

$$\begin{aligned} F(X) &= \text{Hom}_{\mathcal{LR}_k}(\varinjlim_{\alpha \in A} {}^{\mathcal{L}}R_\alpha, X) = \varprojlim_{\alpha \in A} \text{Hom}_{\mathcal{LR}_k}(R_\alpha, X) = \\ &= \varprojlim_{\alpha \in A} \text{Hom}_{\mathcal{LR}_C}({}^{\mathcal{L}}C \widehat{\otimes}_k^u R_\alpha, X) = \text{Hom}_{\mathcal{LR}_C}(\varinjlim_{\alpha \in A} {}^{\mathcal{L}}C \widehat{\otimes}_k^u R_\alpha, X). \end{aligned}$$

□

**Remark 8.5.** This shows that, for any  $C$  in  $\mathcal{RR}_k$ , the functor

$$(8.5.1) \quad \begin{aligned} (-)_C^{\text{ind}} : \mathcal{IRR}_k &\longrightarrow \mathcal{IRR}_C \\ \varinjlim_{\alpha \in A} {}^{\mathcal{L}}R_\alpha &\longmapsto \varinjlim_{\alpha \in A} {}^{\mathcal{L}}(R_\alpha)_C^u \end{aligned}$$

is well defined.

**Corollary 8.6.** *We assume here that  $K$  is a non-archimedean field and that  $k = K^\circ$ . We have*

1. *Let  $M$  and  $N$  be pseudobanach  $K$ -spaces with gauges  $\mathcal{G}$  and  $\mathcal{H}$ , respectively, then  $M \widehat{\otimes}_k^c N$  is a pseudobanach  $K$ -space with gauge  $\mathcal{G} \otimes \mathcal{H}$  consisting of the family of the open sub-objects  $P \widehat{\otimes}_k^u Q$ , for  $P \in \mathcal{G}$  and  $Q \in \mathcal{H}$ .*
2. *If  $K$  is non-trivially valued the category  $\mathcal{PB}_k$  is equivalent to the category  $\text{Ban}_K$  of  $K$ -Banach spaces and continuous  $K$ -linear maps. The tensor product  $M \widehat{\otimes}_k^c N$  corresponds to both the (separated complete) projective and injective tensor products  $M \otimes_{K, \pi} N = M \otimes_{K, \iota} N$  of [12, §17 B].*
3. *If  $K$  is trivially valued so that  $K^\circ = K$ ,  $\mathcal{PB}_K$  is the full subcategory of  $\mathcal{LM}_K$  consisting of  $K$ -vector spaces equipped with the discrete topology and  $-\widehat{\otimes}_K^c - = - \otimes_K -$ .*

## 9 Rings of pm-type

We recall that a subset  $T$  of a topological ring  $R$  is *bounded* if, for any neighborhood  $U$  of 0 in  $R$ , there exists a neighborhood  $V$  of 0 in  $R$  such that  $VT \subset U$ . An element  $x \in R$  is *power bounded* if the set  $T_x = \{1, x, x^2, \dots\}$  is bounded. If  $R$  is a  $k$ -ring and the topology of  $R$  is  $k$ -linear, then  $T$  is bounded if and only if the  $k$ -sub-module of  $R$  generated by  $T$  is bounded. Under the same assumptions, an element  $x \in R$  is power bounded if and only if the  $k$ -sub-ring  $k[x]$  of  $R$  is bounded. For an object  $R$  of  $\mathcal{LR}_k$  we denote by  $R^\circ$  the subset of  $R$  consisting of power bounded elements.

**Lemma 9.1.** *For any object  $R$  of  $\mathcal{LR}_k$ ,  $R^\circ$  is a subring of  $R$ .*

*Proof.* Let  $\mathcal{P}(R)$  be a fundamental system of open  $k$ -submodules of  $R$ , and let  $x, y \in R^\circ$ . We show that both  $T_{x+y}$  and  $T_{xy}$  are bounded. For any  $U \in \mathcal{P}(R)$ , let  $V \in \mathcal{P}(R)$  be such that  $Vk[y] \subset U$ . Let then  $W \in \mathcal{P}(R)$  be such that  $Wk[x] \subset V$ . Then  $Wk[x]k[y] \subset U$ . So,  $k[x]k[y]$  is bounded. The  $k$ -sub-module of  $R$  generated by  $k[x]k[y]$  contains both  $T_{x+y}$  and  $T_{xy}$ . □

**Definition 9.2.** We say that an object  $R$  of  $\mathcal{LR}_k$  is of pm-type if  $R^\circ$  is an open subring of  $R$  and its subspace topology is  $R^\circ$ -linear.

**Remark 9.3.** Notice that, if  $R$  is of pm-type, then  $R^\circ$  is an object of  $\mathcal{RR}_k$  and it is the (unique) maximal open subring  $A$  of  $R$  such that the subspace topology of  $A$  is  $A$ -linear.

**Lemma 9.4.** Let  $R$  be an object of  $\mathcal{LR}_k$ . For any open subring  $A$  of  $R$  the condition that the subspace topology of  $A$  be  $A$ -linear is equivalent to  $A$  being bounded.

*Proof.* In fact, if the subspace topology on  $A$  is  $A$ -linear and  $U$  is any neighborhood of 0 in  $R$ , then there exists an open ideal  $J$  of  $A$  such that  $J \subset U \cap A$ , hence  $JA \subset U$ . Conversely, if  $A$  is bounded and  $U$  is any neighborhood of 0 in  $A$ , there exists a neighborhood  $V$  of 0 in  $A$  such that  $VA \subset U$ . But then  $VA$  is an ideal of  $A$  and a fundamental system of open  $k$ -submodules of  $A$  may be assumed to consist of open ideals of  $A$ .  $\square$

We conclude

**Corollary 9.5.** An object  $R$  of  $\mathcal{LR}_k$  is of pm-type if and only if  $R^\circ$  is an open bounded  $k$ -subring of  $R$ .

## 10 Multivalued rings

We now specialize the definitions of the previous sections to the classical case of topologies defined by a family of semivaluations.

We assume in this section that  $K = (K, v)$  is a non-archimedean field and that  $k = K^\circ$ .

**Definition 10.1.** A semivaluation on a ring  $R$  is a map  $w : R \rightarrow \mathbb{R} \cup \{+\infty\}$  such that

$$w(0) = +\infty, \quad w(x+y) \geq \min(w(x), w(y)), \quad w(xy) \geq w(x) + w(y),$$

for any  $x, y \in R$ ;  $w$  is separated (resp. power-multiplicative, resp. multiplicative) if  $w(x) \neq +\infty$  for  $x \neq 0$  (resp.  $w(x^n) = nw(x)$ , for any  $x \in R$  and  $n \in \mathbb{Z}$ , resp.  $w(xy) = w(x) + w(y)$ , for any  $x, y \in R$ ). A multiplicative semivaluation is called a pseudovaluation, and a valuation if moreover it is separated. A  $k$ -Banach ring is an object of  $\mathcal{LR}_k$  whose topology is induced by a single semivaluation  $w$  extending the valuation  $v$ . Then  $w$  is necessarily separated. A morphism of  $k$ -Banach rings is a continuous  $k$ -algebra morphism, i.e. a morphism in the category  $\mathcal{LR}_k$ .

We denote by  $w^{\text{sp}}$  the spectral valuation associated to  $w$ , that is

$$(10.1.1) \quad w^{\text{sp}}(f) = \lim_n \frac{1}{n} w(f^n) = \sup_n \frac{1}{n} w(f^n).$$

Then  $w^{\text{sp}}$  is in fact a power multiplicative semivaluation on  $R$  and

$$R^\circ = \{x \in R \mid w^{\text{sp}}(x) \geq 0\}.$$

So, a  $k$ -Banach ring  $R$  for the semivaluation  $w$  is of pm-type if and only if  $w$  and  $w^{\text{sp}}$  induce the same topology on  $R$ , or, equivalently, if its topology can be defined by a power-multiplicative semivaluation  $w = w^{\text{sp}}$ . If  $(R_1, w_1), \dots, (R_N, w_N)$  are  $k$ -Banach rings, so is

$$R := (R_1, w_1) \widehat{\otimes}_k^c \dots \widehat{\otimes}_k^c (R_N, w_N).$$

The proof is essentially identical to the one of [12, Lemma 17.2]. More precisely,  $R$  is the separated completion of  $R_1 \otimes_k \dots \otimes_k R_N$  in the product semivaluation  $w$  defined, for any  $y \in R_1 \otimes_k \dots \otimes_k R_N$  by

$$(10.1.2) \quad w(y) = \sup_{j=1, \dots, M} \min_{i=1, \dots, N} w_1(y_{j,1}) + \dots + w_N(y_{j,N})$$

where the supremum is taken over all representations

$$y = \sum_{j=1}^M y_{j,1} \otimes \cdots \otimes y_{j,N}$$

with  $y_{j,i} \in R_i$ , for  $i = 1, \dots, N$ .

A *multivalued  $k$ -ring* is an object  $R$  of  $\mathcal{LR}_k$  whose topology is induced by a family of semivaluations  $\{w_r\}_{r \in T}$ . We write  $R = (R, \{w_r\}_{r \in T})$ . We are especially interested in the case when  $k = K^\circ$ , for a non-archimedean field  $(K, v_K)$  (possibly trivially valued) and the semivaluations  $w_r$ , for  $r \in T$ , induce  $v = v_K$  on  $k$ .

**Remark 10.2.** let  $(K, v_K)$  be a non-trivially valued non-archimedean field. Then a commutative  $K$ -Banach algebra in the classical sense is a  $K^\circ$ -Banach ring  $(A, w)$  for which the scalar product  $K^\circ \times A \rightarrow A$  extends to a structure of  $K$ -vector space on  $A$ . For any commutative  $K$ -Banach algebra  $A$ , one defines classically a subring  $A\{T\}$  of  $A[[T]]$  as the subset of power series  $\sum_{i=0}^{+\infty} a_i T^i$  such that  $\lim_{i \rightarrow +\infty} a_i = 0$ . Then  $A\{T\}$  equipped with the semivaluation  $w_T$  such that

$$w_T\left(\sum_{i=0}^{+\infty} a_i T^i\right) = \inf_{i \geq 0} w(a_i),$$

is a  $K$ -Banach algebra. The semivaluation  $w_T$  is called the *Gauss valuation*. This operation, and terminology, can be iterated in more variables. It is the construction used in [3, 2.1.7].

## 11 Pseudobanach algebras

**Definition 11.1.** A pseudobanach  $k$ -ring is any object  $A$  of  $\mathcal{LR}_k \cap \mathcal{PB}_k$  which admits a gauge  $\mathcal{G}$  such that one element  $R$  of  $\mathcal{G}$  is a  $k$ -subring of  $A$ . A pseudobanach  $k$ -ring is of pm-type if it is of pm-type as an object of  $\mathcal{LR}_k$ . We denote by  $\mathcal{PBA}_k$  (resp.  $\mathcal{UPBA}_k$ ) the full subcategory of  $\mathcal{LR}_k$  whose objects are pseudobanach  $k$ -rings (resp. of pm-type).

**Remark 11.2.** Let  $A$  be a pseudobanach  $k$ -ring. Let  $\mathcal{G}(A)$  be a gauge of  $A$  such that  $R \in \mathcal{G}(A)$  is a subring of  $A$ . The subring  $A^\circ$  of  $A$  is open since it contains  $R$ . So, a pseudobanach  $k$ -ring  $A$  is of pm-type if and only if  $A^\circ$  is bounded. In that case, we may assume that  $A^\circ \in \mathcal{G}$ . Conversely, if  $A^\circ \in \mathcal{G}$ ,  $A$  is of pm-type.

**Remark 11.3.** For  $A, B$  in  $\mathcal{PBA}_k$  the object  $A \hat{\otimes}_k^c B$  of  $\mathcal{PB}_k$  is canonically equipped with a structure of a pseudobanach  $k$ -ring via the structure of  $1$  of Lemma 6.2

Let  $A$  be an object of  $\mathcal{PBA}_k$  with gauge  $\mathcal{G}(A)$  where  $R \in \mathcal{G}(A)$  is a subring of  $A$ . We observe that a power series

$$\sum_{\underline{u} \in \mathbb{Z}_{\geq 0}^n} a_{\underline{u}} \underline{T}^{\underline{u}} \in A[[\underline{T}]] = A[[T_1, \dots, T_n]]$$

is restricted [5, Chap. III, §4, n. 2, Def. 2 p. 253] i.e. is such that for any  $P \subset R$ ,  $P \in \mathcal{G}(A)$ ,  $a_{\underline{u}} \in P$  for almost all  $\underline{u} \in \mathbb{Z}_{\geq 0}^n$ , if and only if  $a_{\underline{u}} \rightarrow 0$  as  $|\underline{u}| := \sum_{i=1}^n u_i \rightarrow +\infty$ . It follows from the definition of a  $k$ -gauge that the product of two restricted power series in  $A[[\underline{T}]]$  is restricted. We denote by  $A\{\underline{T}\}$  the subring of  $A[[\underline{T}]]$  consisting of restricted power series. The family of  $k$ -submodules of  $A\{\underline{T}\}$

$$(11.3.1) \quad U\{\underline{T}\} = \left\{ \sum_{\underline{u} \in \mathbb{Z}_{\geq 0}^n} a_{\underline{u}} \underline{T}^{\underline{u}} \in A\{\underline{T}\} \mid a_{\underline{u}} \in U, \forall \underline{u} \in \mathbb{Z}_{\geq 0}^n \right\}$$

for  $U \in \mathcal{G}(A)$ , is a  $k$ -gauge in  $A\{\underline{T}\}$ . Obviously  $R\{\underline{T}\} \in \mathcal{G}(A\{\underline{T}\})$  is a subring of  $A\{\underline{T}\}$ . It then follows that  $A\{\underline{T}\}$  is an object of  $\mathcal{PBA}_k$

**Definition 11.4.** Let  $A$  be an object of  $\mathcal{PBA}_k$  with gauge  $\mathcal{G}(A)$  where  $R \in \mathcal{G}(A)$  is a ring. Then we regard the ring of restricted power series with coefficients in  $A$  as an object  $A\{\underline{T}\}$  of  $\mathcal{PBA}_k$ , with gauge  $\mathcal{G}(A\{\underline{T}\})$  and  $R\{\underline{T}\} \in \mathcal{G}(A\{\underline{T}\})$  an open subring of  $A\{\underline{T}\}$ .

**Lemma 11.5.** Let  $A$  be an object of  $\mathcal{PBA}_k$ . Then

$$A\{\underline{T}\} = A\widehat{\otimes}_k^c k\{\underline{T}\}.$$

If  $K$  is a non-archimedean field,  $k = K^\circ$ , and  $A$  is a commutative  $K$ -Banach algebra identified with an object of  $\mathcal{PBA}_k$ ,  $A\{\underline{T}\}$  is a commutative  $K$ -Banach algebra and

$$A\{\underline{T}\} = A\widehat{\otimes}_{K,\pi} K\{T\} = A\widehat{\otimes}_{K,\iota} K\{T\}$$

where  $-\widehat{\otimes}_{K,\pi}-$  (resp.  $-\widehat{\otimes}_{K,\iota}-$ ) is the (separated complete) projective (resp. injective) tensor product of [12]. If the topology of  $A$  (resp.  $K\{\underline{T}\}$ ) is induced by a semivaluation  $w$  (resp. by the Gauss semivaluation) the topology of  $A\{\underline{T}\}$  is induced by the product semivaluation (10.1.2) which coincides with the Gauss valuation of  $A\{\underline{T}\}$  of Remark 10.2.

**Example 11.6.** We observe that Definition 11.5 generalizes both the classical definition of ring of restricted power series with coefficients in a linearly topologized ring of [5, Chap. III, §4, n. 2, p. 252-259] and the definition of an  $A$ -affinoid algebra  $A\{T\}$  in the sense of [3, 2.1.7], when  $A$  itself is a commutative  $K$ -Banach algebra.

**Lemma 11.7.** For  $A, B$  in  $\mathcal{UPBA}_k$ ,  $A\widehat{\otimes}_k^c B$  is in  $\mathcal{UPBA}_k$ .

*Proof.* We already pointed out in Remark 11.3 that  $A\widehat{\otimes}_k^c B$  is in  $\mathcal{PBA}_k$  and then, by Remark 11.2,  $(A\widehat{\otimes}_k^c B)^\circ$  is open. (On the other hand the latter fact also follows from  $A^\circ \widehat{\otimes}_k^u B^\circ \subset (A\widehat{\otimes}_k^c B)^\circ$ .) Let  $\mathcal{G}(A)$  (resp.  $\mathcal{G}(B)$ ) be a gauge for  $A$  (resp.  $B$ ) containing an element  $R$  (resp.  $S$ ) which is a subring of  $A$  (resp.  $B$ ). Assume that  $(A\widehat{\otimes}_k^c B)^\circ$  is unbounded. Then, for any open ideal  $I$  of  $k$ , there exists  $x_I \otimes y_I \in (A\widehat{\otimes}_k^c B)^\circ$  such that  $Ix_I \not\subset R$  or  $Iy_I \not\subset S$ . But  $x_I \in A^\circ$  and  $y_I \in B^\circ$ , so this would violate the boundedness of either  $A^\circ$  or  $B^\circ$ , absurd.  $\square$

**Definition 11.8.** Let  $K$  be a non-archimedean field and let  $k = K^\circ$ . A pseudobanach  $K$ -algebra is any object of  $\mathcal{PBA}_k$  which is a  $K$ -vector space. We denote by  $\mathcal{BanAlg}_K$  (resp.  $\mathcal{UBanAlg}_K$ ) the full subcategory of  $\mathcal{LR}_k$  whose objects are pseudobanach  $K$ -algebras (resp. of pm-type).

**Example 11.9.** If  $K$  is non-trivially valued, then the category  $\mathcal{BanAlg}_K$  is equivalent to the commonly used category of  $K$ -Banach algebras, with the only *caveat* that morphisms are simply continuous  $K$ -linear morphisms. The category  $\mathcal{UBanAlg}_K$  is equivalent to the category of  $K$ -Banach algebras of pm-type and continuous  $K$ -algebra morphisms, considered by Fontaine [7].

It follows from Proposition 7.6 that

**Corollary 11.10.** Let  $\kappa$  be an object of  $\mathcal{RR}_k$ . The base-change functor  $A \mapsto (A)_{/\kappa}^c$ , induces additive functors of monoidal categories

$$(\mathcal{PBA}_k, \widehat{\otimes}_k^c) \rightarrow (\mathcal{PBA}_\kappa, \widehat{\otimes}_\kappa^c) \text{ and } (\mathcal{UPBA}_k, \widehat{\otimes}_k^c) \rightarrow (\mathcal{UPBA}_\kappa, \widehat{\otimes}_\kappa^c).$$

Assume in particular  $L/K$  is an extension of non-archimedean valued fields. Then, for  $k = K^\circ$  and  $\kappa = L^\circ$ , the base-change functor  $A \mapsto A_{/\kappa}$  induces  $K$ -linear functors of monoidal categories

$$(\mathcal{BanAlg}_K, \widehat{\otimes}_{K^\circ}^c) \rightarrow (\mathcal{BanAlg}_L, \widehat{\otimes}_{L^\circ}^c) \text{ and } (\mathcal{UBanAlg}_K, \widehat{\otimes}_{K^\circ}^c) \rightarrow (\mathcal{UBanAlg}_L, \widehat{\otimes}_{L^\circ}^c).$$

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